Irrationality proofs for zeta values and dinner parties

Francis Brown, IHÉS-CNRS

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Part I

History
Recall the Riemann zeta values

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Euler proved that $$\zeta(2) = \frac{\pi^2}{6}$$ and more generally

$$\zeta(2n) = -\frac{B_{2n} (2\pi i)^{2n}}{2 (2n)!} \quad \text{for } n \geq 1$$

where $$B_m$$ is the $$m^{\text{th}}$$ Bernoulli number.
Zeta values and Euler’s theorem

Recall the Riemann zeta values

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**Folklore conjecture**

The odd Riemann zeta values \( \zeta(3), \zeta(5), \zeta(7), \ldots \) are algebraically independent over \( \mathbb{Q}[\pi] \).

Very little is known.
A nearly complete list of qualitative known results:

1. The number $\pi$ is transcendental. In particular the even values $\zeta(2n)$ are irrational.

2. The number $\zeta(3)$ is irrational.

3. The vector space spanned by odd zeta values is infinite-dimensional: $\text{dim}_\mathbb{Q} \langle 1, \ldots, \zeta(2n+1), \ldots \rangle = \infty.$

4. One out of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

It is not known whether $\zeta(5) \notin \mathbb{Q}$, or $1, \zeta(2), \zeta(3)$ are linearly independent over $\mathbb{Q}$, nor is it known if $\zeta(3) \notin \pi^3 \mathbb{Q}$. 
A nearly complete list of qualitative known results:

1. (Lindemann 1882). The number $\pi$ is transcendental. In particular the even values $\zeta(2n)$ are irrational.

2. (Apéry 1979). The number $\zeta(3)$ is irrational.

3. (Rivoal and Ball-Rivoal, 2000). The vector space spanned by odd zeta values is infinite-dimensional: $\dim_{\mathbb{Q}} \langle 1, \ldots, \zeta(2n+1), \ldots \rangle_{\mathbb{Q}} = \infty$.

4. (Zudilin, 2001). One out of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. It is not known whether $\zeta(5) \notin \mathbb{Q}$, or $1, \zeta(2), \zeta(3)$ are linearly independent over $\mathbb{Q}$, nor is it known if $\zeta(3)/\pi \notin \mathbb{Q}$. 


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Suppose that we can construct sequences of pairs of rational numbers \( a_n, b_n \) with the following properties:

1. There is a small number \( 0 < \varepsilon < 1 \) such that

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0 < |a_n \alpha - b_n| < \varepsilon^n
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for all sufficiently large \( n \).
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   for all sufficiently large \( n \).

2. Let \( d_n \in \mathbb{N} \) be the common denominator of \( a_n, b_n \):
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   d_n a_n \in \mathbb{Z} \quad d_n b_n \in \mathbb{Z}
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   Assume that \( d_n < D^n \) for some \( D \in \mathbb{R} \).
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3. $D$ is not too big:

$$D\varepsilon < 1$$
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Suppose that we can construct sequences of pairs of rational numbers \( a_n, b_n \) with the following properties:

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There is no integer \( n \) such that \( 0 < n < 1 \)

We only need to construct *small linear forms* in 1 and \( \alpha \) whose denominators are not too big.
Proof (by contradiction). Suppose that $\alpha$ is rational, $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q > 0$. Assumption (1) then becomes

$$0 < \left| a_n \frac{p}{q} - b_n \right| < \varepsilon^n$$

for large $n$

By multiplying through by $q$ and $d_n$, we obtain

$$0 < \left| d_n a_n p - d_n b_n q \right| < q d_n \varepsilon^n < q D^n \varepsilon^n$$

Since by assumption (3) $D \varepsilon < 1$, the right-hand side tends to zero. Thus we can find a large $n$ such that

$$0 < \left| (d_n a_n) p - (d_n b_n) q \right| < 1$$

But by (2), this is an integer between 0 and 1, contradiction.
Let us define
\[ f(x) = \frac{x(1-x)}{1+x} \quad \text{and} \quad \omega = \frac{dx}{1+x} \]
Consider the family of integrals
\[ I_n = \int_0^1 f(x)^n \omega \]
By integrating by parts, one can show that
\[ I_n = r_n \log 2 + s_n \]
where \( r_n \in \mathbb{Z} \) is an integer, and \( s_n \in \mathbb{Q} \) with denominator at most
\[ d(n) := \text{lcm} \(1, 2, \ldots, n\) \]
Theorem (corollary of prime number theorem)

\[ d(n) < e^{n(1+\epsilon)} \text{ where } e = 2.7181 \cdots \]
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Finally, \( f(x) \) is positive on the interval (0, 1), and is bounded above by \( |f(x)| \leq \max_{0<x<1} x(1-x) = \frac{1}{4} \). Therefore we have

\[ 0 < |I_n| < 4^{-n} \]
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The irrationality criteria apply to the linear forms \( I_n \), with

\[ \epsilon = \frac{1}{4} , \quad D = e \]

and we check that \( De \sim 0.679 \cdots < 1 \) and hence (3) holds.

Corollary: \( \log 2 \) is irrational
Theorem (corollary of prime number theorem)

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Finally, \( f(x) \) is positive on the interval \((0, 1)\), and is bounded above by \( |f(x)| \leq \max_{0<x<1} x(1-x) = \frac{1}{4} \). Therefore we have

\[ 0 < |l_n| < 4^{-n} \]

The irrationality criteria apply to the linear forms \( l_n \), with

\[ \varepsilon = \frac{1}{4}, \quad D = e \]

and we check that \( De \sim 0.679 \cdots < 1 \) and hence (3) holds.

**Corollary : log 2 is irrational**

The whole difficulty in this game is to find approximations which satisfy the assumptions (1), (2), (3).
Consider the family of integrals in two variables

\[ I_n = \int_{0 \leq x, y \leq 1} f^n \omega, \]

where \( f = \frac{x(1-x)y(1-y)}{1-xy} \) and \( \omega = \frac{dx dy}{1-xy} \)

One can show that there is an \( a_n \in \mathbb{Z}, b_n \in \mathbb{Q} \) such that

\[ I_n = a_n \zeta(2) + b_n \]

where the denominator of \( b_n \) is bounded by \( d(n)^2 \sim e^{2n} \), and

\[ 0 < I_n < \varepsilon_n \]

where \( \varepsilon = \frac{5}{\sqrt{5} - 11} \sqrt{2} \).

The irrationality of \( \zeta(2) \) follows since

\[ 5 \sqrt{5} - 11 \sqrt{2} e^{2n} = 0. \]
Proof of irrationality of $\zeta(2)$ (Apéry, following Beukers)

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where the denominator of $b_n$ is bounded by $d(n)^2 \sim e^{2n}$, and

$$0 < I_n < \varepsilon^n$$

where $\varepsilon = \frac{5\sqrt{5} - 11}{12}$. The irrationality of $\zeta(2)$ follows since

$$\frac{5\sqrt{5} - 11}{12} e^2 = 0.6627 < 1$$
Consider the family of integrals in three variables:

\[ I_n = \int_{0 \leq x, y, z \leq 1} f^n \omega, \]

where \( f = \frac{x(1 - x)y(1 - y)z(1 - z)}{1 - (1 - xy)z} \) and \( \omega = \frac{dx dy dz}{1 - (1 - xy)z} \).
Proof of irrationality of $\zeta(3)$ (Apéry, following Beukers)

Consider the family of integrals in three variables:

$$I_n = \int_{0 \leq x, y, z \leq 1} f^n \omega,$$

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One can show that

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where the denominator of $b_n$ is bounded by $d(n)^3 < e^{3n}$, and

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where $\varepsilon = (\sqrt{2} - 1)^4$. The irrationality of $\zeta(3)$ follows since

$$(\sqrt{2} - 1)^4 e^3 = 0.59126 \ldots < 1$$
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Many people have tried to construct integrals that give linear combinations of 1 and $\zeta(5)$. The last inequality $D\varepsilon < 1$ fails.
Irrationality measures

Let $\alpha \not\in \mathbb{Q}$ be irrational. The irrationality measure $\mu(\alpha)$ is the infimum of the set of real numbers $\nu$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\nu}$$

has only finitely many solutions $p, q \in \mathbb{Z}$.

Necessarily $\mu(\alpha) \geq 2$.

Liouville numbers such as $\alpha = \sum_{k \geq 1} 10^{-k!}$ have $\mu(\alpha) = \infty$.

Roth’s theorem: if $\alpha$ is algebraic irrational, then $\mu(\alpha) = 2$.

The best known bounds are

$$\mu(\zeta(2)) < 5.442 \quad \text{and} \quad \mu(\zeta(3)) < 5.514$$

are due Rhin and Viola by the group method.
The group method

Let $h, i, j, k, l \geq 0$. Dixon in 1905 considered:

$$\int_{0 \leq x, y \leq 1} \frac{x^h(1 - x)^i y^k(1 - y)^j}{(1 - xy)^{i+j-l}} \frac{dx\,dy}{1 - xy}$$

Rhin and Viola (1996): these give linear forms in $1, \zeta(2)$. It has a large symmetry group of order 1440, which enables one to improve estimates of prime factors of denominators.
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Rhin and Viola (2007):

$$\int_{0 \leq x, y, z \leq 1} \frac{x^h (1 - x)^l y^k (1 - y)^s z^j (1 - z)^q}{(1 - (1 - xy)z)^{q+h-r}} \frac{dxdydz}{1 - (1 - xy)z},$$

where $h, j, k, l, q, r, s \geq 0$ subject to the constraints

$$j + q = l + s \quad \text{and} \quad k + r \geq h$$

It gives linear forms in $1, \zeta(3)$ and has group $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \Sigma_5$. 
Nesterenko’s criterion for linear independence

Let $\alpha_1, \ldots, \alpha_r$ be real numbers. Suppose that we have linear forms

$$I_n = a^1_n \alpha_1 + \ldots + a^r_n \alpha_r$$

such that $a^i_n$ are integers and that

$$|a^i_n| \leq \eta^n \quad \text{for all } i, \text{ and large } n$$

$$\lim_{n \to \infty} |I_n|^{1/n} = \varepsilon$$

where $0 < \varepsilon < 1$. Then

$$\dim_{\mathbb{Q}} \langle \alpha_1, \ldots, \alpha_r \rangle > 1 - \frac{\log \varepsilon}{\log \eta}$$
Nesterenko’s criterion for linear independence

Let $\alpha_1, \ldots, \alpha_r$ be real numbers. Suppose that we have linear forms

$$I_n = a_1^n \alpha_1 + \ldots + a_r^n \alpha_r$$

such that $a_n^1$ are integers and that

$$|a_n^i| \leq \eta^n$$

for all $i$, and large $n$

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where $0 < \varepsilon < 1$. Then

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Idea is now to construct linear forms in $1, \zeta(2), \zeta(3), \ldots, \zeta(n)$ and apply the above. Unfortunately, the linear forms are not good enough to prove independence; we already know the subspace

$$\langle 1, \zeta(2), \zeta(4), \ldots, \zeta(2k) \rangle_{\mathbb{Q}}$$

has dimension $k + 1$ by Lindemann. Want to kill $\zeta(2n)$’s.
A breakthrough in 2000 was the introduction of very-well poised hypergeometric series. Fischler (after Zlobin) found the following integral representation for the linear forms of Ball-Rivoal:

\[ \int_{[0,1]^{a-1}} \frac{\prod_{j=1}^{a-1} x_j^r (1 - x_j)^n \, dx_j}{(1 - x_1 x_2 \cdots x_{a-1})^{rn+1} \prod_{2 \leq 2j \leq a-2} (1 - x_1 x_2 \cdots x_{2j})^{n+1}} \]

where \( n \geq 0, \ a \geq 3 \) and \( 1 \leq r < \frac{a}{2} \) are integers.
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where \( n \geq 0,\ a \geq 3 \) and \( 1 \leq r < \frac{a}{2} \) are integers.

These integrals give small linear forms in

- 1, \( \zeta(3), \zeta(5), \ldots, \zeta(a - 1) \) if \( a \) even
- 1, \( \zeta(2), \zeta(4), \ldots, \zeta(a - 1) \) if \( a \) odd
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Applying Nesterenko’s criterion to the first gives: the Ball-Rivoal theorem on odd zeta values. Applying it to the second gives another proof of the transcendence of \( \pi \).
Picard-Fuchs recurrences

The linear forms occurring in Apéry’s proof are of the form

\[ a_n \zeta(3) + b_n \]

where \( a_n \) is the sequence of integers

\[ a_1 = 1, \ a_2 = 5, \ a_3 = 73, \ a_4 = 1445, \ a_5 = 33001 \]

The sequences \( a_n \) and \( b_n \) are solutions to the recurrence relation:

\[
(n + 1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0
\]

It is remarkable that such a recurrence has a solution which are all integers! There are numerous interpretations of this recurrence relation as a Picard-Fuchs equation of a family of varieties. Interesting connections with modular forms. The coefficients satisfy many congruence and super-congruence relations . . .
Part II

Geometry
Let $n \geq 3$. The configuration space of $n$-points in $\mathbb{P}^1$ is

$$C^n = \{(z_1, \ldots, z_n) \in \mathbb{P}^1 : z_i \text{ distinct}\}$$
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The group $\text{PSL}_2$ acts on $\mathbb{P}^1$ by projective transformations

$$z \mapsto \frac{az + b}{cz + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2.$$

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$$\mathcal{M}_{0,n} = C^n / \text{PSL}_2 .$$
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We can always put $z_1 = 0, z_{N-1} = 1, z_N = \infty$. Therefore $\mathcal{M}_{0,n}$ is the complement of hyperplanes

$$\mathcal{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in \mathbb{A}^{n-3} \text{ such that } t_i \neq 0, 1 \text{ and distinct}\}$$
Let $n \geq 3$. The configuration space of $n$-points in $\mathbb{P}^1$ is

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I claim that most (possibly all) known irrationality results for zeta values are related to $\mathcal{M}_{0,n}(\mathbb{R})$. 
Examples

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Here is a picture of $\mathcal{M}_{0,5}$:

![Diagram of $\mathcal{M}_{0,5}$]

The group $\Sigma_n$ acts on $\mathcal{M}_{0,n}$ by permuting the marked points.
Connected components of $\mathcal{M}_{0,n}(\mathbb{R})$

The points of $\mathcal{M}_{0,n}(\mathbb{R})$ are in one-to-one correspondence with $n$ distinct marked points on a circle $\mathbb{R} \cup \{\infty\}$ up to automorphisms.
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The symmetric group $\Sigma_n$ permutes the set of cells $X^{\delta}$. 
A class of integrals

A class of integrals (periods) of $\mathcal{M}_{0,n}$ is given by

$$I = \int_{X^{\delta_0}} \omega$$

where $\omega \in \Omega^{n-3}(\mathcal{M}_{0,n}; \mathbb{Q})$ is a regular algebraic $n-3$-form.
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It is a linear combination of integrals

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**Theorem (B. 2006)**

$I$ is a $\mathbb{Q}$-linear combination of multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}$$

where $n_r \geq 2$ and $n_1 + \ldots + n_r \leq n - 3$. 
A general construction

The proof of the theorem is effective (algorithms by B.-Bogner, E. Panzer). In principle it gives bounds, e.g., on denominators.
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We could go a very long way if one could understand:

**Vanishing problem**

Find conditions on \( f, \omega \) to force certain coefficients \( a_{ni} \) to vanish.
Cohomological interpretation

Let $\overline{M}_{0,n}$ be the Deligne-Mumford-Knudsen compactification. The singularities of $f^n\omega$ define a boundary divisor $A$, the Zariski closure of the boundary of $X^{δ₀}$ defines a boundary divisor $B$.

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Find $A, B \subset \overline{M}_{0,n} \setminus M_{0,n}$ such that

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**Theorem**

This is possible for $n = 5$ (trivial) and $n = 6$ (tricky).

I do not know if it is possible for any $n \geq 7$. 
Part III

Dinner Parties
Consider two dihedral orderings \((\delta, \delta')\) on \(\{1, \ldots, n\}\). They correspond to two connected components on \(\mathcal{M}_{0,n}(\mathbb{R})\).
Cellular integrals

Consider two dihedral orderings \((\delta, \delta')\) on \(\{1, \ldots, n\}\). They correspond to two connected components on \(\mathcal{M}_{0,n}(\mathbb{R})\).

Define an \(n\)-form on the configuration space \(\mathcal{C}^n\) by:

\[
\tilde{\omega}_{\delta'} = \pm \frac{dz_1 \ldots dz_n}{\prod_{i \in \mathbb{Z}/n\mathbb{Z}} (z_{\delta'_i} - z_{\delta'_{i+1}})}
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It is PSL\(_2\)-invariant and descends to a form \(\omega_{\delta'} \in \Omega^{n-3}(\mathcal{M}_{0,n})\).
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Now define a rational function on \(C^n\) by:

\[
\tilde{f}_{\delta/\delta'} = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{z_{\delta_i} - z_{\delta_{i+1}}}{z_{\delta'_i} - z_{\delta'_{i+1}}}.
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Define the basic cellular integrals to be

\[
I_{\delta/\delta'}(N) = \int_{X^\delta} f_{\delta/\delta'}^N \omega_{\delta'} \quad \text{for } N \geq 0
\]
Example:

Let $N = 5$, and $\delta = (1, 2, 3, 4, 5), \delta' = (1, 3, 5, 2, 4)$. Then

$$\tilde{f}_{\delta/\delta'}(z) = \frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_4 - z_5)(z_5 - z_1)}{(z_1 - z_3)(z_3 - z_5)(z_5 - z_2)(z_2 - z_4)(z_4 - z_1)}$$
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Set \( z_1 = 0, z_2 = t_1, z_3 = t_2, z_4 = 1 \) and let \( z_5 \) go to \( \infty \). We get

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f_{\delta/\delta'}(t) = \frac{t_1(t_1 - t_2)(t_2 - 1)}{t_2(1 - t_1)} \quad \text{and} \quad \omega_{\delta'} = \frac{dt_1 dt_2}{t_2(1 - t_1)}
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They give back exactly the Apéry linear forms in $1, \zeta(2)$.

Warning

The integral $I_{\delta/\delta'}(N)$ does not always converge! We want to understand for which $\delta, \delta'$ it converges.
The dinner table problem

Suppose that we have $N$ guests for dinner, sitting on a round table. It is boring to talk to the same person for the whole duration of the meal, so after the main course, we should permute the guests around in such a way that no-one is sitting next to someone they previously sat next to.

The first solution is for $N = 5$, and is unique.
The enumeration of dinner table seating plans was computed by Poulet in 1919. We actually need a variant where consecutive blocks of $k$ guests don’t sit next to each other.
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The dinner table problem is $k = 2$. We need $k = \left\lfloor \frac{N}{2} \right\rfloor$.

This seating plan for 8 guests is bad for us: a block of four consecutive guests 1, 2, 3, 4 (and 5, 6, 7, 8) are sitting together.
Geometric meaning

The domain of integration is simply the cell $X^\delta$. The form $\omega_{\delta'}$ has singularities contained in the boundary of $X^{\delta'}$. The rational function $f_{\delta'/\delta'}$ vanishes along the boundary of $X^\delta$ and has poles along the boundary of $X^{\delta'}$. 

Recall that the symmetric group $\Sigma_n$ acts on $M_0$, $n$. Two pairs of dihedral orderings are equivalent if $(\delta,\delta') \sim (\sigma\delta,\sigma\delta')$ for some $\sigma \in \Sigma_n$. Call the equivalence class a configuration. Equivalent configurations give the same cellular integrals. A configuration $(\delta,\delta')$ is convergent if $I_{N\delta/\delta'}$ is finite for all $N$. We can always assume that $\delta = \delta_0$ from now on.
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The multiple zeta values of sub-maximal weight always vanish.
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**Enumeration of convergent configurations:**

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**Theorem**

For $N = 5, 6$ there is a unique class of convergent configurations. The basic cellular integrals give back exactly Apéry’s proofs of the irrationality of $\zeta(2)$ and $\zeta(3)$, respectively.
Linear forms in multiple zeta values

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Starting with $N = 8$ we find linear forms involving products such as $\zeta(2)\zeta(3)$ as well as $\zeta(5)$. 
Ball-Rivoal’s theorem and Lindemann’s theorem

Theorem

Let $m \geq 3$. The family of convergent configurations $(\delta_0, \pi)$

$$\pi^m_{\text{odd}} = (2m, 2, 2m-1, 3, 2m-2, 4, \ldots, m, 1, m+1)$$

gives Ball-Rivoal’s forms in $1, \zeta(3), \zeta(5), \ldots, \zeta(2m-3)$. The family

$$\pi^m_{\text{even}} = (2m+1, 2, 2m, 3, 2m-1, 4, \ldots, m+2, 1, m+1)$$

gives back their linear forms in $1, \zeta(2), \zeta(4), \ldots, \zeta(2m-2)$. 

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There appears to be a whole zoo of configurations with interesting vanishing properties. For instance, the dual configuration

$$(\pi^m_{odd}, \delta_0) \sim (\delta_0, (\pi^m_{odd})^{-1})$$

yields new linear forms in

$$1, \pi^2, \pi^4, \ldots, \pi^{2m-6}, \zeta_{2m-3}$$

Can one do a $p$-adic or single-valued version to kill the $\pi^{2n}$’s?
Generalised cellular integrals

We can introduce parameters into the cellular integrals by

\[ \tilde{f}_{\delta/\delta'} = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{(z_{\delta_i} - z_{\delta_{i+1}})^{a_{i,i+1}}}{(z_{\delta'_i} - z_{\delta'_{i+1}})^{b_{i,i+1}}} \]

where \( a_{i,i+1}, b_{i,i+1} \) are integers chosen such that the expression is homogeneous in each \( z_i \). Each basic cellular integral on \( \mathcal{M}_{0,n} \) spawns a large family of integrals with \( n \) parameters.
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**Theorem**

The generalised cellular integrals, for \(N = 5\) and \(N = 6\) are equivalent to Rhin and Viola’s integrals for \(\zeta(2)\) and \(\zeta(3)\).
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The generalised cellular integrals, for \(N = 5\) and \(N = 6\) are equivalent to Rhin and Viola’s integrals for \(\zeta(2)\) and \(\zeta(3)\).

The dinner party game generates all irrationality results.
Generalised cellular integrals

We can introduce parameters into the cellular integrals by

\[
\tilde{f}_{\delta/\delta'} = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{(z_{\delta_i} - z_{\delta_{i+1}})^{a_{i,i+1}}}{(z_{\delta'_i} - z_{\delta'_{i+1}})^{b_{i,i+1}}}
\]

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The dinner party game generates all irrationality results.

The generalised integrals for \(\pi^m_{odd}\) give a huge family of integrals that appears to give linear forms in odd zetas, with a rich symmetry group. Can one improve on Ball-Rivoal’s theorem?
Picard-Fuchs recurrences

Every family of basic cellular integrals $I_{\pi}(N)$ satisfies a Picard-Fuchs recurrence equation. Some properties:

1. (Poincaré duality). The family $I_{\pi^\vee}$ of the dual configuration $\pi^\vee$ satisfies the dual (homogeneous) Picard-Fuchs equation.
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$$I_\pi(N) = I_{\pi_1}(N)I_{\pi_2}(N) \quad \text{for all } N \geq 0$$

This gives a partial multiplication law.
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3. (Relations). Sometimes, for non-equivalent $\pi, \pi'$ we have

$$I_\pi(N) = I_{\pi'}(N) \quad \text{for all } N \geq 0$$

When does this happen?