Floer theory in spaces of stable pairs over Riemann surfaces

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Let $Z$ be a closed Riemann surface. Among the gauge-theoretic moduli spaces $M(Z)$ associated with it, those that are intrinsically compact Kähler manifolds include

- $N_{\mathfrak{d}}^b$, the projectively flat connections in a $U(2)$-bundle of odd degree $d$.

Sending a connection $A$ to the holomorphic structure defined by $\overline{\partial}_A$ defines a biholomorphic map to the moduli space $N^{ss}$ of rank 2 semistable vector bundles.

- $V_{L,\tau}$, the space of vortices in a hermitian line bundle $L \rightarrow Z$ of degree $d$:

$$(A, \phi) : \quad \overline{\partial}_A \phi = 0, \quad iF_A + |\phi|^2 \eta = \tau \eta.$$ 

($\eta$ is a fixed area form on $Z$ with $\int_Z \eta = 1$, and $\tau > 0$). $\overline{\partial}_A$ defines a holomorphic structure in $L$ making $\phi$ a holomorphic section, so we get a map

$$V_{L,\tau} \rightarrow \text{Sym}^d Z, \quad [A, \phi] \mapsto \phi^{-1}(0).$$

This map is biholomorphic for $\tau > 2\pi d$. The resulting Kähler form on $\text{Sym}^d(Z)$ lies in a class varying affine-linearly with $\tau$. 

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Floer theory for stable pairs
Rank 2 vortices

- A **holomorphic pair** is a holomorphic vector bundle $V \to Z$, together with a non-trivial holomorphic section $\phi$. Numerical parameters $(r, d) = (\text{rank}, \text{degree})$. $\text{Sym}^d(Z)$ is a fine moduli space of $(1, d)$ holomorphic pairs.

- $V_{E, \tau}$, the space of **vortices** in a hermitian $\mathbb{C}^2$-bundle $E \to Z$:

  $$(A, \phi) : \quad \bar{\partial}_A \phi = 0, \quad iF_A + \frac{1}{2}(\phi \otimes \phi^*) \eta = \frac{1}{2} \tau \eta \text{Id}.$$ 

  $\eta$ is a fixed area form on $Z$, normalized to have total area 1, and $\tau > 0$.

- Bradlow (1990), Bradlow–Daskalopoulos (1993): $V_{E, \tau}$ is a compact Kähler manifold, and the map

  $$V_{L, \tau} \to \{(2, d) \text{ holomorphic pairs})\}, \quad [A, \phi] \mapsto [\bar{\partial}_A, \phi]$$

  is biholomorphic onto the coarse moduli space of $(2, d)$ **$\sigma$-semistable pairs**, $\sigma = \frac{d}{2} - \tau$. 

Stable pairs

- Fix $\sigma > 0$. A $(\text{rank, degree}) = (2, d)$ holomorphic pair $(E, \phi)$ is called $\sigma$-semistable if, for all line bundles $F \subset E$,
  \begin{enumerate}
  \item $\deg F \leq \frac{d}{2} + \sigma$; and moreover
  \item $\deg F \leq \frac{d}{2} - \sigma$ if $\phi \in H^0(F)$.
  \end{enumerate}
It's $\sigma$-stable if we can sharpen $\leq$ to $<$.  

- There are coarse moduli spaces $M_{d,\sigma}$, fine for most $\sigma$. We fix a fiber $\Lambda$ of the determinant submersion $\det: M_{d,\sigma} \to \text{Pic}^d(Z), [E, \phi] \mapsto \wedge^2 E$, to define $M_{\Lambda,\sigma}$.

- Thaddeus (1992) gives a precise and beautiful description of the moduli spaces $M_{\Lambda,\sigma}$ which I’ll review shortly.

- The compact Kähler manifolds $M_{\Lambda,\sigma}$ are the subject of this lecture.
Gauge theory vs. symplectic geometry

- The equations for flat connections and rank 1 vortices are dimensional reductions of equations in 4 dimensions with gauge symmetry: instanton, Seiberg–Witten with a closed, non-exact 2-form perturbation.
- The rank 2 vortex equations are (almost) the dimensional reductions of 4-dimensional non-abelian SW equations studied by Feehan–Leness and others.
- Instanton, SW invariants of 3- and 4-manifolds containing $Z$ are intimately related to symplectic topology of $N^b(Z)$ and $\text{Sym}^d Z$ respectively, in particular to Lagrangian submanifolds and holomorphic curves.
- When $d$ is even, the moduli space $N^b$ of projectively flat connections is singular, and problematic for Floer theory. Instanton Floer theory is also hard to set up beyond the case of homology 3-spheres, because of problems with singularities.
- **Aspiration:** use a space of stable pairs $M_{\Lambda,\sigma}$ (with $d$ even) as a substitute for $N^b$, and construct 3-manifold invariants via Floer theory in $M_{\Lambda,\sigma}$. 

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Floer theory for stable pairs
Structure of $M_{\Lambda, \sigma}$ (Thaddeus)

$(E, \phi)$ $\sigma$-semistable: for all line bundles $F \subset E$, $\deg F \leq \frac{d}{2} + \sigma$, and moreover $\deg F \leq \frac{d}{2} - \sigma$ if $\phi \in H^0(F)$.

Take $d \geq 0$ even.

- $\phi$ is always a section of some line bundle $F_\phi \subset E$ (of maximal degree). Since $\deg F_\phi \geq 0$, we have $\sigma \leq d/2$.
- We get a sequence of non-empty moduli spaces $M_i = M_{\Lambda, (d/2) - i - \epsilon}$, for $i = 0, 1, \ldots \frac{d}{2} - 1$ and $\epsilon \in (0, 1)$.
- In $M_0$, we must have $\deg F_\phi \leq 0$, so $(F_\phi, \phi)$ is a deg 0 rank 1 holomorphic pair (must be $(\mathcal{O}_Z, 1)$), while $E$ is an extension of $\mathcal{O}_Z$ by $\Lambda$. Must be non-split, but that’s the only constraint.

We get

$$M_0 = \mathbb{P}H^1(\Lambda^{-1}) = \mathbb{P}H^0(K_Z\Lambda)^*.$$  

- In $M_1$, $F_\phi$ could have degree 1; the deg 1 holo. pairs form $Z$. In fact, $M_1$ is the blow-up of $M_0$ along $Z$ embedded via $|K_Z\Lambda|$.
- $M_{i+1}$ is a flip of $M_i$ for $i > 0$.
- All are smooth projective of dimension $d + g - 2$; simply connected; Picard rank 2 for $i > 0$. 

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We’re most interested in the last in the sequence of flips, $M_{\text{top}} = M_{d/2}^{−1}$. That is, $\sigma \in (0, 1)$; $(E, \phi)$ is $\sigma$-semistable if $E$ is a semistable bundle and $F_\phi$ does not destabilize $E$.

There’s an Abel–Jacobi map

$$M_{\text{top}} \to N_{\Lambda}^{ss}, \quad [E, \phi] \mapsto [E]$$

whose fibers are the projective spaces $\mathbb{P}H^0(E)$.

For $d > 2g − 2$, Abel–Jacobi is surjective and we think of it as a sort of ‘resolution’, in that $N_{d}^{ss}$ is singular (of dim $3g − 3$) while $M_{\text{top}}$ is non-singular (of dimension $g + d − 2$).

We’ll focus on $M_{\text{top}}$ because it’s closest to the world of stable bundles and flat connections.
Recall that Heegaard Floer theory is based on $\text{Sym}^d Z$ with $d = g(Z)$. The reason for $d = g$ is that a handlebody $U$ bounding $Z$ defines interesting Lagrangian submanifolds of $\text{Sym}^g Z$ specifically.

These Lagrangians (which are tori) can be constructed explicitly:

1. explicitly: the product of $g$ disjoint circles that bound in $U$;
2. implicitly: as limits of solutions to the SW equations on the cylindrical completion of in $U \setminus B^3$, with a Taubes-type perturbation; or as iterated vanishing cycles of degenerations.

The analogous degree for rank 2 stable pairs (and the rank 2 SW equations over handlebodies) turns out to be $d = 2g + 2$.

From now, on $M_Z$ denotes $M_{\text{top}}$ for a fixed determinant $\Lambda$ of degree $2g + 2$.

It is smooth projective of dimension $3g$.

**Fortuitous observation:** $M_Z$ is Fano! Specific to $(d, \sigma) = (2g + 2, \text{small})$. 

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A non-abelian Heegaard Floer theory??

\[ M_Z = M_{top} \text{ for } d = 2g + 2. \text{ Smooth projective, Fano of dim } 3g. \]

- That \( M_Z \) is Fano implies that any pair of simply connected embedded Lagrangians have well-defined Floer cohomology.

**Conjecture:** In degree \( 2g + 2 \), a handlebody \( U \) bounding \( Z \) defines an embedded Lagrangian submanifold \( L_U \subset M_Z \), diffeomorphic to \( (S^3)^g \).

- If true, these could be used to form a Heegaard-Floer type theory based on Floer cohomology for the pair of Lagrangians coming from a Heegaard splitting.

- When \( g = 1 \) (so \( d = 4 \)), \( M_Z \) is the blow-up of \( \mathbb{C}P^3 \) along \( Z \), embedded via a degree 4 linear system. The conjecture is true here (the Lagrangians are vanishing cycles for a Lefschetz pencil with \( M_Z \) as fiber). We haven’t yet managed to prove it for \( g > 1 \).
Fibered 3-manifolds

- Gauge theory also has a symplectic interpretation on fibered 3-manifolds $Y^3 \to S^1$.

- Let $Z$ be the fiber, $\phi$ the monodromy. For any $d \geq 0$, the symplectic fixed point Floer homology group, for the symplectic action of $\phi$ on $\text{Sym}^d Z$, is isomorphic to a summand in the monopole Floer homology of $Y$ (with suitable perturbations). The summand corresponds to a subset of the Spin$^c$-structures.

- This suggests that the fixed point Floer homology for the action of $\phi$ on rank 2 stable pairs is also worth exploring. All degrees $d$ are of interest in this setting, but since we are interested in the Fukaya category of $M_Z$ we shall also focus on the (related) fixed point Floer homology for $M_Z$.
Set up for fixed point Floer homology

Equivalent data:

- \((M, \omega, \phi)\) cpt. manifold, symplectic form, symplectic automorphism

\[\uparrow\]

- \((T \to S^1, \Omega)\) proper fiber bundle, closed fiberwise-symp. 2-form.

\((M, \omega, \phi) \to\) mapping torus \((p_\phi : T_\phi \to S^1, \omega_\phi)\)

fiber, monodromy \(\leftarrow\) \((p : T \to S^1, \Omega)\)

Here \(T_\phi = (M \times \mathbb{R})/(x, t) \sim (\phi(x), t + 1)\) and \(p_\phi^* \omega_\phi = \omega\).

Monodromy is for the symplectic connection \(H^\Omega = (\ker Dp)^\Omega\).

- Fixed points \(\leftrightarrow\) horizontal sections
- Adding closed 2-form \(\eta\), zero on fibers, \((T \to S^1, \Omega + \eta)\) gives symp. isotopy \((M, \omega, \{\phi_t\}_{t \in [0,1]}\).

Flux \(\phi_t \in H^1(M; \mathbb{R})\) lies in \(\text{im}(1 - \phi_0^*)\) iff \(\eta\) exact on \(T\).
Fixed-point Floer homology

- To each monotone symplectic automorphism \( \phi \in \text{Aut}(M, \omega) \),

\[
[\omega_\phi] = \lambda c_1(T^{\text{vert}} T_\phi) \in H^2(T_\phi; \mathbb{R}), \quad \lambda > 0,
\]

we can attach its fixed-point Floer homology \( HF(M; \phi) \).

- Finitely generated, \((\mathbb{Z}/2)\)-graded abelian group;
  Euler characteristic = Lefschetz number \( \Lambda_\phi \).
  Module over quantum cohomology \( QH^*(M) = (H^*(M; \mathbb{Z}), \ast) \).

- Invariant under isotopies \( \{ \phi_t \} \) with flux in \( \text{im}(1 - \phi_0^*)|_{H^1(M; \mathbb{R})} \).

- If \( \phi \) has non-degenerate fixed points,

\[
HF(M; \phi) = H_\ast(CF_\ast(\phi), \partial J), \quad CF_\ast(\phi) = \mathbb{Z}^{\text{fix} \phi},
\]

graded by Lefschetz signs.

- Matrix entries \( \langle \partial J x_-, x_+ \rangle \) count \( J \)-holomorphic sections \( u \) of \( T_\phi \times \mathbb{R} \to S^1 \times \mathbb{R} \) with \( \lim_{t \to \pm \infty} u(\cdot, t) = x_\pm \) (where \( J \) is a suitable translation-invariant almost complex structure).
Monodromy acting on stable pair spaces

- Let $Z$ be a closed, connected, oriented surface and $\Lambda \to Z$ a complex line bundle. There’s a central extension of the mapping class group $\Gamma = \pi_0\text{Diff}^+(Z)$,

$$1 \to H^1(Z;\mathbb{Z}) \to \tilde{\Gamma} \to \Gamma \to 1.$$ 

$\tilde{\Gamma} := \{ (\phi, \tilde{\phi}) \text{ up to isotopy} \}: \phi \in \text{Diff}^+(Z) \text{ and } \tilde{\phi}: \Lambda \xrightarrow{\sim} \phi^*\Lambda.$

- Fix a complex structure in $Z$ and a holomorphic structure in $\Lambda$. Let $M = M_{\Lambda,\sigma}$ be the space of $\sigma$-stable pairs over $Z$ with determinant $\Lambda$. Let $\omega_M$ be a Kähler form.

- There’s a homomorphism

$$\mu: \tilde{\Gamma} \to (\text{Aut } /\text{Ham})(M,\omega):$$

constructed as follows:

- Build from $\tilde{\phi}$ a line bundle $\Lambda_{\tilde{\phi}} \to T_{\tilde{\phi}}$. Choose fiberwise complex structure in $T_{\tilde{\phi}}$, holomorphic structure in $\Lambda_{\tilde{\phi}}$.

- Associated bundle $M$-bundle $M_{\phi} \to S^1$ has $H^2(M_{\phi}) = H^2(M)$.

- Choose any closed, fiberwise Kähler 2-form $\Omega$ in $M_{\phi}$ extending $\omega_M$. Take monodromy.
Stable pair Floer homology

- When \( \deg \Lambda = 2g_Z + 2 \), and \( \omega_M \) an anticanonical Kähler form, \( \Phi := \mu(\tilde{\phi}) \) is a monotone symplectic automorphism. Define

\[
HSP(\tilde{\phi}) := HF(M, \Phi),
\]

a \( \mathbb{Z}/2 \)-graded abelian group, module over \( \mathbb{Q}H^*(M) \).

- It breaks into generalized eigenspaces for \( c_1(M) \star \cdot \):

\[
HSP(\tilde{\phi}) \otimes \mathbb{C} = \bigoplus_{\lambda} HSP(\tilde{\phi}; \mathbb{C})_{\lambda},
\]

Non-zero summands can only be for \( \lambda \) zero or an eigenvalue of \( c_1(M) \star \cdot \) acting on \( \mathbb{Q}H^*(M; \mathbb{C}) \).
The genus 1 case: quantum cohomology

When $Z$ is an elliptic curve, $M_Z = \text{Bl}_Z(\mathbb{C}P^3)$. Here $Z = Q_0 \cap Q_1$ (complete intersection of quadric surfaces).

**Proposition**

The generalized eigenspace decomposition for $c_1(M_Z) \star \cdot$ acting on $\text{QH}^*(M_Z)$ is as follows:

\[
\text{QH}^*(M_Z) \otimes \mathbb{C} = \text{QH}_{-1} (\text{dim 4}) \\
\oplus \text{QH}_0 \oplus \text{QH}_8 \oplus \text{QH}_{-4-4i} \oplus \text{QH}_{-4+4i} \text{ (lines)}.
\]

There is a $\mathbb{C}$-algebra isomorphism $\text{QH}^*_{-1} \cong H^*(Z; \mathbb{C})$. Thus, as algebras,

\[
\text{QH}^*(M_Z) \otimes \mathbb{C} = H^*(Z; \mathbb{C}) \oplus \mathbb{C}^4.
\]

Proof is by direct calculation.

The four simple eigenvalues agree with critical values of the Hori–Vafa–Givental mirror superpotential.
The genus 1 case: Floer homology

Theorem (A. Lee–P.)

For $Z$ of genus 1 and $\tilde{\phi} \in \tilde{\Gamma}$ homologically non-degenerate, there are isomorphisms of $(\mathbb{Z}/2)$-graded abelian groups

$$HSP(\tilde{\phi}) \cong HSP(\tilde{\phi})_{-1} \oplus \mathbb{Z}_{\text{even}}$$

$$HSP(\tilde{\phi})_{-1} \cong HF(Z; \phi).$$

Homologically non-degenerate means $\phi^* - 1$ invertible on $H^1(Z; \mathbb{Q})$. Equivalently, $\phi$ is not a power of a Dehn twist. Under this condition:

$$HSP(\tilde{\phi}) \cong HSP(\tilde{\phi})_{-1} \oplus \mathbb{Z}_{\text{even}}$$

$$HSP(\tilde{\phi})_{-1} \cong HF(Z; \phi).$$
The genus 1 case: Floer homology

**Theorem (A. Lee–P.)**

For $Z$ of genus 1 and $\tilde{\phi} \in \tilde{\Gamma}$ homologically non-degenerate, there are isomorphisms of $(\mathbb{Z}/2)$-graded abelian groups

$$HSP(\tilde{\phi}) \cong HSP(\tilde{\phi})_{-1} \oplus \mathbb{Z}_{\text{even}}^4$$

$$HSP(\tilde{\phi})_{-1} \cong HF(Z; \phi).$$

**Notes on $HF(Z; \phi)$:**

- In the homologically non-degenerate case, it’s $\mathbb{Z}^F$. Here $F = (\phi^* - 1)^{-1}(L)/L$ where $L = H^1(Z; \mathbb{Z})$.
- It lives in degree $d$, where $(-1)^d = \text{sign det}(\phi^* - 1)$.
- **Y.-J. Lee–Taubes:** it’s SW monopole Floer homology for $T_\phi$, summed over Spin$^c$-structures $s$ with $c_1(s)[Z] = [2]$, with negative monotone perturbations.
Proving the first clause: \( HSP(\widetilde{\phi}) = HSP(\widetilde{\phi})^{-1} \oplus \mathbb{Z}^4 \)

- A certain lift \( \widetilde{\tau} \in \widetilde{\Gamma} \) of a Dehn twist \( \tau \in \Gamma \) acts on \( M_Z \) by a Dehn twist around a Lagrangian 3-sphere \( V \).

- The count \( m^0(V) \) of Maslov 2 holomorphic discs on \( V \) is necessarily an eigenvalue of \( c_1(M_Z) \star \cdot \). By an argument of I. Smith, \( m^0(V) = -1 \).

- It follows that \( c_1 + I \) is nilpotent on \( HF(V, L) \) for any other monotone Lagrangian \( L \) with \( m^0(V) = -1 \).

- There is an exact triangle

\[
\cdots \to HF(V, \mu(\widetilde{\phi})(V)) \to HSP(\widetilde{\phi}) \to HSP(\widetilde{\tau} \circ \widetilde{\phi}) \to \cdots .
\]

Taking the sum of all the generalized eigenspaces for eigenvalues \( \lambda \neq -1 \), the sequence remains exact but the first term dies.

- Hence for any composite of lifted Dehn twists, this part of \( HSP \) is the same as for the identity: \( \mathbb{Z}^4_{even} \).
\[ HSP(\widetilde{\phi})_{-1} \cong HF(Z; \phi): \text{how we don’t prove it} \]

- I. Smith uses Lagrangian correspondences to embed the Fukaya category of a genus \( g > 1 \) surface into that of the blow-up of \( \mathbb{C}P^{2g+1} \) along an intersection of two quadrics.

- There’s still a Lagrangian correspondence from \( Z \) to \( M_Z \), but it appears that it does not induce a functor

\[ \mathcal{F}(Z) \rightarrow \mathcal{F}(M_Z)_{-1} \]

because of holomorphic discs attached to the correspondence. (Perhaps the obstruction can be cancelled by a bulk deformation of \( \mathcal{F}(Z) \)—cf. ideas in a slightly different context of Fukaya.)
Let $\tilde{X}$ be the blow-up of a complex manifold $X$ along a complex-codimension 2 submanifold $Y$.

Let $f$ be a Morse function on $X$, generic in that it has no critical points on $Y$ while $f|_Y$ is Morse. Its pullback $\tilde{f}$ to $\tilde{X}$ is again Morse.

On the exceptional divisor $E = \mathbb{P}N_{Y/X}$, $\tilde{f}$ has exactly one critical point $\lambda_y$ over each critical point $y \in \text{crit}(f|_Y)$. Namely, $\lambda_y$ is the unique complex line in $(N_{Y/X})_y$ contained in $\ker(D_y f : N_y Y \to \mathbb{R})$.

We have $\text{ind}_{\tilde{X}}(\lambda_y) = \text{ind}_Y(y) + 2$.

Hence if $f$ and $f|_Y$ are perfect Morse functions, so is $\tilde{f}$. 

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Floer theory for stable pairs
Computing the chain complex

- The mapping torus $T_\Phi$ of $\Phi = \mu(\tilde{\phi})$ is the ‘family blow-up’, relative to $S^1$, of $\mathbb{C}P^3 \times S^1$ along $T_\phi$.

- Using an explicit model for the symplectic blow-up, we can arrange that the fixed points of the symplectic monodromy are in bijection with those of $\phi$, together with 4 coming from a hamiltonian automorphism of $\mathbb{C}P^3$. (This is much like the toy model.)

- Take $\phi$ to be the action of a homologically non-degenerate element in $SL_2(\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$. When $\det(\phi^* - 1)|_{H^1(\mathbb{Z};\mathbb{Q})} > 0$, all fixed points are even, so the Floer differential is trivial and we’re done.

- When $\det(\phi^* - 1)|_{H^1(\mathbb{Z};\mathbb{Q})} < 0$, there are exactly 4 even fixed points. The differential on the Floer complex must be trivial so as to have rank $\text{HSP}_{\text{even}} \geq 4$.

- I hid a snag with this argument...
Continuity of Floer homology

- Snag: The explicit model is for low-weight blow-ups (i.e., cohomology classes \([\omega_t] = 4h - t[E]\) for \(t\) small), while Floer homology was defined for an anticanonical symplectic form \(\omega_1\).
- We choose to handle this using continuity of Floer homology.
- Fixed-point Floer homology can be ‘classically’ defined for automorphisms of any compact symplectic 6-manifold: the continuity maps used to prove invariance are not available because of bubbling.
- **Continuity principle**, Y.-J. Lee, Usher: In a family \((M, \omega_t, \Phi_t)_{t\in[0,1]}\) where all \(HF(\Phi_t)\) are well-defined over the same field, rank \(HF(\Phi_t)\) is constant provided that the symplectic action \(A_t\) on the period group \(P\) varies in a simple way: \(A_t = f(t)A_{t_0}\), where \(f(t) \geq 0\).
- Use this principle to see that we can deform from low-weight blow-up forms \(\omega_t\) to an anticanonical form \(\omega_1\).
- Avoid bubbling in this borderline case by using Kähler forms and keeping the chosen complex structure unchanged (up to small perturbations) through the deformation.
Higher genus? $M_Z$ contains an interesting codimension $g$ submanifold: extensions

$$0 \to F \to E \to \Lambda F_\phi \to 0$$

where $(F, \phi)$ is a holomorphic pair of the highest allowed degree, $g$. It's a $\mathbb{P}^g$-bundle over $\text{Sym}^g Z$. **Guess:** $HSP(\phi)$ contains $g$ copies of the fixed-point Floer cohomology action on $\text{Sym}^g Z$, coming from fixed points here. This locus hints at a relationship with Heegaard Floer theory.

We can also obtain results on Lagrangian Floer cohomology for $g = 1$, and again see a relation with SW theory. The results for $g = 1$ are consistent with the notion that $M_Z$ is a space of interest with respect to Floer-theoretic invariants of 3-manifolds.

The critical next step is the construction of embedded Lagrangian submanifolds from handlebodies. I'm working on this, by means of degenerations of $Z$.

If it can be done one gets a well-defined Floer cohomology group from any Heegaard splitting. If it is stabilization-invariant, one gets a 3-manifold invariant.