Computing maps between Fukaya categories via Morse trees

Nathaniel Bottman
Princeton/IAS
September 2017
§1: context
The Fukaya category of a symplectic manifold
The Fukaya category of a symplectic manifold

A symplectic manifold is \((M^{2n}, \omega)\), with \(\omega \in \Omega^2(M)\) closed, \(\omega^n\) nonvanishing.

Eg: \((M, \omega) = (\text{real surface, area form})\).
The Fukaya category of a symplectic manifold

A symplectic manifold is \((M^{2n}, \omega)\), with \(\omega \in \Omega^2(M)\) closed, \(\omega^n\) nonvanishing.

Eg: \((M, \omega) = (\text{real surface, area form})\).

A Lagrangian is \(L^n \subset M^{2n}\) with \(\omega |_L = 0\).

Eg: \(L = \text{embedded curve}\).
The Fukaya category of a symplectic manifold
The Fukaya category of a symplectic manifold

Roughly, the **Fukaya category** $\text{Fuk}(M, \omega)$ has:

- objects are Lagrangians $L \subset M$,
- morphisms are $\text{hom}(L, K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$. 
The Fukaya category of a symplectic manifold

Roughly, the **Fukaya category** $\text{Fuk}(M, \omega)$ has:

- **objects** are Lagrangians $L \subset M$,
- **morphisms** are $\text{hom}(L, K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$.

Composition? Fix $a \in L^2 \cap L^1$, $b \in L^1 \cap L^0$; coefficient of $c \in L^2 \cap L^0$ in $a \circ b$ is a count of rigid pseudoholomorphic triangles:
The Fukaya category of a symplectic manifold

Roughly, the **Fukaya category** $\text{Fuk}(M, \omega)$ has:
- objects are Lagrangians $L \subset M$,
- morphisms are $\text{hom}(L, K) := \mathbb{K}\langle p \rangle_{p \in L \cap K}$.

Composition? Fix $a \in L^2 \cap L^1$, $b \in L^1 \cap L^0$; coefficient of $c \in L^2 \cap L^0$ in $a \circ b$ is a count of rigid pseudoholomorphic triangles:

...composition **not** associative!

but can make into an $A_\infty$-category by counting rigid polygons.
Functoriality for Fuk?
Idea (Bottman, building on MWW+Weinstein): build an $(A_\infty, 2)$-category, $\text{Symp}$, whose objects are $M$'s and hom$(M, N) := \text{Fuk}(M^- \times N)$. E.g., need:

$\text{Fuk}(M_0^- \times M_1) \otimes \text{Fuk}(M_1^- \times M_2) \rightarrow \text{Fuk}(M_0^- \times M_2)$
Idea (Bottman, building on MWW+Weinstein): build an $(A_\infty, 2)$-category, $\text{Symp}$, whose objects are $M$’s and $\text{hom}(M, N) := \text{Fuk}(M^- \times N)$. E.g., need:

$$\text{Fuk}(M_0^- \times M_1) \otimes \text{Fuk}(M_1^- \times M_2) \to \text{Fuk}(M_0^- \times M_2)$$

Do so by counting **witch balls** — pseudoholomorphic maps from the colored patches to symplectic manifolds, with “seam conditions” given by Lagrangian correspondences.
§2: 2-associahedra
Defining the 2-associahedra

To understand the algebraic structure of $\text{Symp}$, need to understand the degenerations that can take place in the domain moduli space $\overline{2\mathcal{M}_n}$, where:

$$2\mathcal{M}_n := \left\{ \begin{array}{c} (x_1, \ldots, x_r) \in \mathbb{R}^r \\
(y_{11}, \ldots, y_{1n_1}) \in \mathbb{R}^{n_1} \\
\vdots \\
(y_{r1}, \ldots, y_{rn_r}) \in \mathbb{R}^{n_r} \\
\end{array} \begin{array}{c} x_1 < \cdots < x_r \\
y_{11} < \cdots < y_{1n_1} \\
\vdots \\
y_{r1} < \cdots < y_{rn_r} \\
\end{array} \right\} / \mathbb{R}^2 \times \mathbb{R}_{>0}$$
Theorem (B, arXiv: 1709.00119): For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r$, the 2-associahedron $W_{\mathbf{n}}$ is a poset which is an abstract polytope.
Theorem (B, 2017): The 2-associahedra form a relative 2-operad over the associahedra.

Corollary: Can finally define the notion of $(A_\infty, 2)$-category!
Theorem (B, 2017): The 2-associahedra form a relative 2-operad over the associahedra.

Corollary: Can finally define the notion of $(A_\infty, 2)$-category!

\[ W_2 \times W_{100} \times K_3 W_{200} \leftrightarrow W_{300} \]
§3: computation via Morse trees?
Polygons in $T^*B$
Polygons in $T^*B$

Fix a metric $g$ on $B$; get $g : TB \to T^*B$.

Identify $T(T^*B) \simeq TB \otimes T^*B$ and define:

$J_\epsilon \in \text{End}(T(T^*B))$, $J_\epsilon := \begin{pmatrix} 0 & \epsilon g^{-1} \\ -\epsilon^{-1} g & 0 \end{pmatrix}$
Polygons in $T^*B$

Fix a metric $g$ on $B$; get $g : TB \to T^*B$.

Identify $T(T^*B) \simeq TB \oplus T^*B$ and define:

$$J_\epsilon \in \text{End}(T(T^*B)), \quad J_\epsilon := \begin{pmatrix} 0 & \epsilon g^{-1} \\ -\epsilon^{-1}g & 0 \end{pmatrix}$$

Question (Fukaya—Oh): Characterize $J_\epsilon$-hol. strips (polygons) with bdry on $\Gamma(df)$'s?
Polygons in $T^* B$

For small $\epsilon$, fibers of $T^* B$ look small and strips become linear in the fibers:

$$(p, df_0(p))$$

$$(p, df_1(p))$$
Polygons in $T^*B$

For small $\epsilon$, fibers of $T^*B$ look small and strips become linear in the fibers:

\[(p, df_0(p))\]

\[(p, df_1(p))\]

strip is $J_\epsilon$-holomorphic $\implies$ \[
\dot{p}(t) = \epsilon \, d(f_1 - f_0)(p(t))
\]

$\implies$ \[
p(\epsilon^{-1}t) \text{ Morse flowline}
\]

$\implies$ \[
\left( \text{hom}(\Gamma(df_0), \Gamma(df_1)), \mu^1 \right) \simeq (CM(f_1 - f_0), d_{\text{Morse}})
\]
Polygons in $T^* B$

And similarly for polygons in $T^* B$: 

\[ \Gamma(df_0), \Gamma(df_1), \Gamma(df_2), \Gamma(df_3) \]

\[ f_1 - f_0, f_2 - f_1, f_3 - f_2, f_3 - f_0 \]
...how about witch balls?

**Question:** How about witch balls in cotangent bundles?
thanks!