On Voevodsky’s Univalence principle

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Joint work with
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Axiomatic Homotopy Theory

J.H.C. Whitehead (1950): The ultimate aim of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that analytic is equivalent to pure projective geometry.

Traditional axiomatic systems in homotopy theory:
- Triangulated categories [Verdier 1963];
- Homotopical algebra [Quillen 1967];
- Derivators [Grothendieck 1984]

New axiomatic systems:
- Higher toposes [Rezk, Lurie, ....];
- Homotopy type theory [Voevodsky, Awodey & Warren, ....];
- Cubical type theory [Coquand & collaborators].
A univalent type theory (UTT) is obtained by adding the univalence principle to Martin-Löf type theory (MLTT).

The goal of Voevodsky’s Univalent Foundation Program is to apply UTT to the foundation for mathematics.

Another goal is to develop computerised proof assistants based on UTT.

Important advances were made since the Special Year on UFP at the IAS in 2012-13.

The development of Cubical Type Theory (CTT) by Thierry Coquand and collaborators is a huge step forward.

But many basic questions remain to be solved.
Goals of my talk

- To present UTT semantically
- To discuss Voevodsky’s univalence principle
- To describe applications

Fact 1: EVERY MATHEMATICIAN IS USING TYPE THEORY WITHOUT BEING AWARE OF IT

Fact 2: Voevodsky’s univalence principle is to type theory what the induction principle is to Peano arithmetic.

Fact 3: Univalence ⇒ Descent

Fact 4: Descent ⇒ Generalised Blakers – Massey theorem ⇒ Goodwillie’s Calculus
Overview

The notion of tribe is a gateway to type theory.

\[ \text{Tribe} \Rightarrow \text{Type Theory} \]

Benedikt Ahrens will give a more formal presentation of Type Theory in this conference

1. Introduction to TT via the notion of tribe
2. What is univalence?
3. Univalence and descent
4. The Blakers-Massey theorem
5. The generalised Blaker-Massey theorem
6. Goodwillie’s Calculus
7. Higher sheaves
The syntaxic category of type theory

Theorem
(Gambino & Garner, Shulman) The syntaxic category of type theory is tribe.

Theorem
(Shulman, J.) Every tribe is a Brown's fibration category
A clan is a category equipped with a class of carrable maps called fibrations.

Recall that a map $p : X \rightarrow B$ in a category $\mathcal{C}$ is said to be carrable if the fiber product of $p$ with any map $f : A \rightarrow B$ exists,

$$
\begin{array}{c}
\begin{array}{c}
A \times_B X \quad \pi_2 \\
\downarrow \pi_1 \\
A \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\quad \pi_2 \\
\downarrow p \\
X \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\quad f \\
\downarrow \\
B \\
\end{array}
\end{array}
\end{array}
$$

The projection $\pi_1$ is called the base change of $p$ along $f$. 
The notion of clans

Definition
A clan is a category $C$ equipped with a class $\mathcal{F}$ of maps called fibrations satisfying the following conditions:

- every isomorphism is a fibration;
- the composite of two fibrations is a fibration;
- every fibration is carrable and the base change of a fibration along any map is a fibration;
- $C$ has a terminal object $1$ and the unique map $X \to 1$ is a fibration for every object $X \in C$.

Definition
A homomorphism of clans $F : C \to C'$ is a functor which preserves:

- fibrations and base changes of fibrations;
- terminal objects.
Anodyne maps

Definition
A map $u : A \to B$ in a clan $C$ is said to be anodyne if it has the left lifting property with respect to every fibration $f : X \to Y$.

This means that every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{anodyne~u} & & \downarrow{f~fibration} \\
B & \xrightarrow{b} & Y
\end{array}
\]

has a diagonal filler $d : B \to X$ ($du = a$ and $fd = b$).
Definition
A tribe $\mathcal{E}$ is a clan in which
- the base change of an anodyne map along a fibration is anodyne;
- every map $f : A \to B$ admits a factorization $f = pu$ with $u$ anodyne and $p$ a fibration:

\[
\begin{array}{c}
E \\
\uparrow u \\
A \quad f \quad \downarrow p \\
\downarrow \uparrow f \\
\downarrow \quad \quad \downarrow \quad \quad B.
\end{array}
\]

Definition
A homomorphism of tribes $F : \mathcal{E} \to \mathcal{E}'$ is a homomorphism of clans which takes anodyne maps to anodyne maps.
The tribe of Kan complexes

Recall that a simplicial set $X$ is said to be a **Kan complex** if every horn $h : \Lambda^k[n] \to X$ has a **filler** $h' : \Delta[n] \to X$.

![Diagram 1]

Recall that a map of simplicial sets $f : X \to Y$ is said to be a **Kan fibration** if every commutative square

![Diagram 2]

has a diagonal filler $h' : \Delta[n] \to Y$. 

---
The tribe of Kan complexes

Let us denote the category of Kan complexes by $\text{Kan}$; it is a full subcategory of the category of simplicial sets.

**Theorem**

*The category $\text{Kan}$ has the structure of a tribe, where a fibration is a Kan fibration.*

**Remark:** a map (between Kan complexes) $u : A \to B$ is anodyne iff it is a monomorphism and a homotopy equivalence (iff it is a strong deformation retract).
Elementary extension

If $A$ is an object in a tribe $\mathcal{E}$, then the full subcategory of $\mathcal{E}/A$ whose objects are fibrations $p : X \to A$ is a tribe $\mathcal{E}(A)$.

By definition, a morphism $f : (X, p) \to (Y, q)$ in $\mathcal{E}(A)$ is a fibration if the map $f : X \to Y$ is a fibration in $\mathcal{E}$.

The tribe $\mathcal{E}(A)$ is an elementary extension of $\mathcal{E}$.

The functor $e_A : \mathcal{E} \to \mathcal{E}(A)$ defined by putting $e_A(X) = (A \times X, \pi_1)$ is a homomorphism of tribes.

If $f : A \to B$ is a map in a tribe $\mathcal{E}$, then the base change functor

$$f^* : \mathcal{E}(B) \to \mathcal{E}(A)$$

is a homomorphism of tribes.
Types and elements

If $\mathcal{E}$ is a tribe, then for every object $A \in \mathcal{E}$ we shall write

$$\mathcal{E} \vdash A : \text{Type} \quad \text{or} \quad \vdash A : \text{Type}$$

and say that $A$ is a type.

If $a : 1 \to A$ is a map in $\mathcal{E}$, we shall write

$$\mathcal{E} \vdash a : A \quad \text{or} \quad \vdash a : A$$

and say that $a$ is an element of type $A$.

Remark: an element $a : A$ is often called a term of type $A$. 
A fibration is a family of objects.

The fiber \( E(a) := p^{-1}(a) \) of a fibration \( p : E \to A \) at an element \( a : A \) is defined by the pullback square

\[
\begin{array}{ccc}
E(a) & \to & E \\
\downarrow & & \downarrow p \\
1 & \overset{a}{\to} & A.
\end{array}
\]

This defines a family of types \((E(a)|a : A)\) indexed by elements \( a : A \).

The object \((E, p)\) of the tribe \( \mathcal{E}(A) \) is called a dependant type in context \( A \). A type theorist will write

\[ x : A \vdash E(x) : Type \quad (1) \]

where \( E(x) \) denotes the fiber of \( p : E \to A \) at a variable element \( x : A \).
Sections of fibration

A section \( s : A \to E \) of a fibration \( p : E \to A \) has a value \( s(a) : E(a) \) for every element \( a : A \).

A type theorist will write:

\[
x : A \vdash s(x) : E(x)
\]  

(2)

\( s(x) \) denotes the value of \( s : A \to E \) at a *variable element* \( x : A \).
Change of parameters

If $f : A \to B$ is a map in a tribe $\mathcal{E}$, then the base change functor

$$f^* : \mathcal{E}(B) \to \mathcal{E}(A)$$

is a homomorphism of tribes.

In type theory, the functor $f^*$ is defined by the operation of substitution: $y := f(x)$

$$y : B \vdash E(y) : \text{Type} \quad \quad \quad y : B \vdash s(y) : E(y)$$

$$x : A \vdash E(f(x)) : \text{Type} \quad \quad \quad x : A \vdash s(f(x)) : E(f(x))$$
In type theory, the elementary extension $e_A : \mathcal{E} \to \mathcal{E}(A)$ is defined by context extension:

\[
\begin{align*}
\vdash B : \text{Type} & \quad \vdash t : B \\
x : A \vdash B : \text{Type} & \quad x : A \vdash t : B
\end{align*}
\]

A map between two types $f : A \to B$ is a variable element $f(x) : B$ indexed by variable $x : A$.

\[
x : A \vdash f(x) : B
\]  \quad (3)
**Σ-formation rule**

If $A$ is an object in a tribe $\mathcal{E}$, then the functor $e_A : \mathcal{E} \to \mathcal{E}(A)$ has a left adjoint $\Sigma_A : \mathcal{E}(A) \to \mathcal{E}$ defined by putting $\Sigma_A(E, p) = E$.

$$
\frac{x : A \vdash E(x) : Type}{\vdash \sum_{x : A} E(x) : Type}
$$

More, generally if $f : A \to B$ is a fibration, then the base change functor $f^* : \mathcal{E}(B) \to \mathcal{E}(A)$ has a left adjoint $\Sigma_f : \mathcal{E}(A) \to \mathcal{E}(B)$.

$$
\frac{x : A \vdash E(x) : Type}{y : B \vdash \sum_{x : f^{-1}(y)} E(x) : Type}
$$
Internal products

A tribe $\mathcal{E}$ has **internal products** if the base change functor $f^* : \mathcal{E}(B) \to \mathcal{E}(A)$ has a right adjoint $\Pi_f : \mathcal{E}(A) \to \mathcal{E}(B)$ for every fibration $f : A \to B$ and

- the functor $\Pi_f$ takes anodyne maps to anodyne maps;
- the Beck-Chevalley condition holds.

In particular, the functor $e_A : \mathcal{E} \to \mathcal{E}(A)$ has a right adjoint

$$\Pi_A : \mathcal{E}(A) \to \mathcal{E}$$

for every object $A \in \mathcal{E}$.

$$\begin{align*}
x : A &\vdash E(x) : Type \\
\vdash \prod_{x:A} E(x) : Type
\end{align*}$$
Path object

Let $A$ be an object in a tribe $\mathcal{E}$.

A **path object** for $A$ is obtained by factoring the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,

\[
\begin{array}{ccc}
PA & \xrightarrow{(s,t)} & A \times A \\
\downarrow r & & \downarrow \Delta \\
A & \xrightarrow{\Delta} & A \times A.
\end{array}
\]

A **homotopy** $h : f \sim g$ between two maps $f, g : A \to B$ is a map $h : A \to PB$ such that $sh = f$ and $th = g$.

The homotopy relation $f \sim g$ is a congruence on the arrows of $\mathcal{E}$. 
The homotopy category

Let $\mathcal{E}$ be a tribe

The **homotopy category** of $\mathcal{E}$ is the quotient of $\mathcal{E}$ by the homotopy relation $\sim$.

$$ho(\mathcal{E}) := \mathcal{E}/\sim$$

A map $f : X \to Y$ in $\mathcal{E}$ is called a **homotopy equivalence** if it is invertible in $ho(\mathcal{E})$.

Every anodyne map is a homotopy equivalence.

An object $X$ is **contractible** if the map $X \to 1$ is a homotopy equivalence.
Identity type

In Martin-Löf type theory, there is a type constructor which associates to every type $A$ a dependant type

$$x : A, y : A \vdash \text{Id}_A(x, y) : \text{Type}$$

called the identity type of $A$,

An element $p : \text{Id}_A(x, y)$ is regarded as a proof that $x =_A y$.

There is a reflexivity term

$$x : A \vdash r(x) : \text{Id}_A(x, x)$$

which proves that $x =_A x$. 
The identity type is a path object

Let us put

\[ Id_A = \sum_{x:A} \sum_{y:A} Id_A(x, y) \]

(Awodey & Warren, Voevodsky). The factorisation \( \Delta = (s, t)r \) a path object for \( A \).

\[ \begin{array}{ccc}
Id_A & \xrightarrow{r} & (s,t) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\Delta} & A \times A
\end{array} \]

A proof \( p : Id_A(x, y) \) is a homotopy \( p : x \rightsquigarrow y \).
Tribes are fibration categories

Theorem

(Shulman, J.) A tribe is a Brown fibration category.

By definition, a Brown fibration category $\mathcal{E}$ is equipped with a class of fibrations $\mathcal{F}$ and a class of equivalences $\mathcal{W}$.

- $(\mathcal{E}, \mathcal{F})$ is a clan;
- the base change of an equivalence along a fibration is an equivalence;
- every map $f : A \to B$ admits a factorization $f = pw$ with $w$ an equivalence and $p$ a fibration;
- $\mathcal{W}$ satisfies 6-for-2.
HOTT in action

Definition (Voevodsky) We say that an object \( A \) is **contractible** if the object

\[
\text{IsCont}(A) := \sum_{y:A} \prod_{x:A} \text{Id}_A(x, y)
\]

has an element \( p : \text{IsCont}(A) \).

This may be compared with

\[
\text{IsSingleton}(A) := (\exists y \in A)(\forall x \in A) x = y
\]

(the Curry-Howard correspondance).
The object \textit{IsEquiv}(f) \\

Let \( f : A \to B \) be a map in a tribe.

The \textit{homotopy fiber} of \( f \) at \( y : B \) is defined by putting

\[
\text{Fib}(f)(y) := \sum_{x : A} \text{Id}_B(f(x), y)
\]

\textbf{Definition} \\
(Voevodsky) A map \( f : A \to B \) is a \textbf{homotopy equivalence} if the object

\[
\text{IsEquiv}(f) := \prod_{y : B} \text{IsCont}(\text{Fib}(f)(y))
\]

has an element \( p : \text{IsEquiv}(f) \).
The object $Eq(A, B)$

Let $A$ and $B$ be two objects of a tribe. The object of maps $A \to B$ is defined by putting

$$[A, B] := \prod_{x : A} B$$

Definition

(Voevodsky) The object $Eq(A, B)$ of homotopy equivalences $A \simeq B$ is defined by putting

$$Eq(A, B) := \sum_{f : [A, B]} IsEquiv(f)$$
For every fibration $p : E \to A$ let us put

$$Eq_{A \times A}(E) = \sum_{x:A} \sum_{y:A} Eq(E(x), E(y))$$

We then have a factorisation

$$\begin{array}{c}
\text{Eq}_{A \times A}(E) \\
\downarrow \quad (p_1, p_2) \\
A \quad \Delta \\
\downarrow \quad \downarrow \\
A \times A
\end{array}$$

where $u(x) : Eq(E(x), E(x))$ represents the identity map $E(x) \to E(x)$ for every $x : A$. 
Univalent fibration

The commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{u} & Eq_{A \times A}(E) \\
\downarrow r & & \downarrow (p_1, p_2) \\
Id_A & \xrightarrow{(s, t)} & A \times A
\end{array}
\]

has a diagonal filler \( \gamma : Id_A \to Eq_{A \times A}(E) \), since \( r \) is anodyne and \((p_1, p_2)\) is a fibration.

**Definition**

(Voevodsky) We say that a fibration \( E \to A \) is **univalent** if the map \( \gamma : A \to Eq_{A \times A}(E) \) is a homotopy equivalence.

This means that the map

\[
\gamma(x, y) : Id_A(x, y) \to Eq(E(x), E(y))
\]

is a homotopy equivalence for every \( x, y : A \).
Small fibrations

Let $\mathcal{E} = (\mathcal{E}, \mathcal{F})$ be a tribe.

We say that $\mathcal{F}' \subseteq \mathcal{F}$ is a class of **small fibrations** if the following conditions hold:

- every isomorphism is a small fibration;
- if $f : A \to B$ and $g : B \to C$ are fibrations and $g$ is small, then $f$ is small if and only if $gf$ is small;
- if $f : A \to B$ is a homotopy equivalence and $g : B \to C$ and $gf$ are fibrations, then $g$ is small if and only if $gf$ is small;
- the base change of a small fibration along any map is small;
- the internal product of a small fibration along a small fibration is small.

There is a notion of $\kappa$-small fibration in the tribe of Kan complexes $\text{Kan}$ for every strongly inaccessible cardinal $\kappa$. 
Recall that a commutative square

\[
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & & \downarrow g \\
A & \longrightarrow & B
\end{array}
\]

is said to be **homotopy cartesian** if the map \( F \to A \times_B E' \) in the diagram

\[
\begin{array}{ccc}
F & \longrightarrow & A \times_B E' & \longrightarrow & E' \\
\downarrow & & \downarrow & & \downarrow g' \\
A & \longrightarrow & B & \longrightarrow & B
\end{array}
\]

is a homotopy equivalence, where \( g = g'w : E \to E' \to B \) is a factorisation of \( g \) as a homotopy equivalence followed by a fibration.
Universal fibration

A small fibration \( q : E \to U \) is said to be **universal** if for every small fibration \( p : E \to A \) there exists a homotopy cartesian square

\[
\begin{array}{ccc}
E & \xrightarrow{\phi'} & E \\
\downarrow p & & \downarrow q \\
A & \xrightarrow{\phi} & U
\end{array}
\]

and if moreover the pair \((\phi, \phi')\) is homotopy unique.

Voevodsky: The uniqueness condition holds if the fibration \( q : E \to U \) is univalent.

The object \((U, q)\) is said to be a **universe**.

Martin-Löf type theory (MLTT) only have pseudo-universes (they are not univalent).
Theorem

(Voevodsky) The tribe of Kan complexes \( \text{Kan} \) has a universe \( \mathbb{E}_\kappa \to \mathbb{U}_\kappa \) for each strongly inaccessible cardinal \( \kappa \).

A *Univalent Type Theory* (UTT) is an extension of Martin-Löf type theory (MLTT) in which semi-universes are univalent: the univalence principle holds.

Cubical Type Theory (CTT) extends MLTT with a new formalism, with a richer notion of context. The univalence principle *can be proved* in CTT.
Descent

Recall that a commutative square of simplicial sets

\[
\begin{array}{ccc}
A & \overset{v}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
B & \longrightarrow & F
\end{array}
\]

is said to be **homotopy cocartesian** if the map \(B \sqcup_A E' \to F\) in the commutative diagram

\[
\begin{array}{ccc}
A & \overset{v'}{\longrightarrow} & E' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \sqcup_A E' \longrightarrow F
\end{array}
\]

is a homotopy equivalence, where \(v = wv' : A \to E' \to E\) is a factorisation of \(v\) as a cofibration \(v'\) followed by homotopy equivalence \(w\).
Descent

A cube $C : [1]^3 \to \mathcal{E}$, when viewed from above, becomes a square $C' : [1]^2 \to \mathcal{E}^{[1]}$ in the arrow category of $\mathcal{E}$.

$$\begin{align*}
  f \quad \beta \\
  \alpha & \downarrow & \downarrow \delta \\
  g & \gamma & \downarrow \\
  \quad & \quad & k
\end{align*}$$

(4)

An edge of $C'$ is a square in $\mathcal{E}$.

Theorem

[Rezk] (Descent for cubes) Suppose that the square $C' : [1]^2 \to \Delta \textbf{Set}^{[1]}$ is homotopy cocartesian. If the squares $\alpha$ and $\beta$ of $C'$ are homotopy cartesian, then so are the squares $\delta$ and $\gamma$. 
Univalence $\implies$ Descent

The proof is left as an exercise to the reader. Hint: we can suppose that the maps $f$, $g$, $h$ and $k$ are Kan fibrations; we then use the following diagram

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\downarrow & \downarrow h & \downarrow k \\
\delta & \phi_1 & \phi_2 \\
\gamma & Q & Q \\
\end{array}
\]

where $Q : E \to U$ is a univalent Kan fibration, and where $\phi_1$ and $\phi_2$ are classifying $f$ and $g$ respectively.
Model topos

Definition
[Rezk] A combinatorial model category $\mathcal{E}$ is said to be a **model topos** if it is Quillen equivalent to a left exact Bousfield localisation of a model category $[C, \Delta \text{Set}]$ equipped with the projective model structure.

Theorem
[Rezk] A combinatorial model category $\mathcal{E}$ is a model topos if and only if the following two conditions hold:

1. The base change functor $f^* : \mathcal{E}/B \to \mathcal{E}/A$ preserves homotopy colimits for every map between fibrant objects $f : A \to B$;
2. The descent principle holds (for cubes).
From type theory to higher toposes

ML type theory
\[ \xymatrix{ \text{syntaxic category} \\ \text{tribes} \\ \text{lcc quasicategories} } \]

Cubical type theory
\[ \xymatrix{ \text{syntaxic category} \\ \text{Cubical tribes} \\ \text{\(\infty\) -- toposes} } \]

- syntaxic category [Gambino & Garner]
- localisation [Kapulkin & Szumilo]
- elementary \(\infty\)-topos [Lurie, Shulman]
The Blakers-Massey theorem

Recall that a simplical set $X$ is said to be \((-1)\)-\textit{connected} if it is non-empty.

A simplical set $X$ is said to be \(0\)-\textit{connected} if it is connected.

If $n \geq 1$, a simplical set $X$ is said to be \(n\)-\textit{connected} if it is connected and $\pi_k(X,x) = 0$ for every $1 \leq k \leq n$ and any $x \in X$.

We shall say that a map of simplicial set $f : X \to Y$ is \(n\)-\textit{connected} if its homotopy fibers are \(n\)-connected.

Warning: a \(n\)-connected map as defined here is often said to be \((n + 1)\)-\textit{connected} in the literature.
The Blakers-Massey theorem

Theorem (Blakers-Massey) Suppose that we have a homotopy pushout square of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \downarrow{} \\
B & \rightarrow & D
\end{array}
\]

in which \(f\) is \(m\)-connected and \(g\) is \(n\)-connected. Then the canonical map \((f, g) : A \rightarrow B \times^h_D C\) to the homotopy pullback is \((m + n)\)-connected.
The Blakers-Massey

The BM theorem implies the Freudenthal suspension theorem:

If $X$ is a pointed $n$-connected space, then the canonical map $X \to \Omega \Sigma X$ is $2n$-connected:

$$
\begin{array}{ccc}
X & \xrightarrow{n\text{-}con} & CX \\
\downarrow & & \downarrow \\
CX & \xrightarrow{n\text{-}con} & \Sigma X
\end{array}
$$
A new proof of Blakers-Massey

A type theoretic proof of the BM theorem was found by Favonia & Eric Finster & Dan Licata & Peter Lumsdane [FFLL]. The proof is NEW!

It was reformulated in the language of model categories by Charles Rezk.

The proof uses descent for cube.

The BM theorem and its proof were generalised by Mathieu Anel & Georg Biedermann & Eric Finster & J. [ABFJ].
The generalised Blakers-Massey theorem

In the Generalised Blakers-Massey theorem the class of \( n \)-connected maps is replaced by the left class \( \mathcal{L} \) of a modality in an \( \infty \)-topos.

The theory of modalities in HOTT was developed by Egbert Rijke & Michael Shulman & Bas Spitter.

By definition, a \textbf{modality} in an \( \infty \)-topos \( \mathcal{E} \) is a homotopy factorisation system \( (\mathcal{L}, \mathcal{R}) \) in which the left class \( \mathcal{L} \) is closed under base changes.

For example, \( \mathcal{L} \) can be the class of effective epimorphisms in \( \mathcal{E} \) and \( \mathcal{R} \) is the class of monomorphisms.

More generally, \( \mathcal{L} \) can be the class of \( n \)-connected maps in \( \mathcal{E} \) and \( \mathcal{R} \) the class of \( n \)-truncated maps.
The generalised Blakers-Massey Theorem

Recall that the **pushout product** $f \Box g$ of two maps $f : A' \to A$ and $g : B' \to B$ in an $\infty$-topos $\mathcal{E}$ is the map

$$f \Box g : (A' \times B) \sqcup_{A' \times B'} (A \times B') \to A \times B$$

obtained from the commutative square

$$
\begin{array}{ccc}
A' \times B' & \xrightarrow{f \times B'} & A \times B' \\
\downarrow A' \times g & & \downarrow A \times g \\
A' \times B & \xrightarrow{f \times B} & A \times B
\end{array}
$$
The generalised Blakers-Massey Theorem

Recall that the diagonal of a map $f : A \to B$ in an $\infty$-topos $\mathcal{E}$ is the map

$$\Delta(f) : A \to A \times_B A$$

**Theorem**

*(Generalised BM theorem)* [ABFJ] Let $(\mathcal{L}, \mathcal{R})$ be a modality in an $\infty$-topos $\mathcal{E}$ and let

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow \\
X & \xrightarrow{} & W
\end{array}
\]

be a pushout square in $\mathcal{E}$. If $\Delta(f) \Box \Delta(g) \in \mathcal{L}$, then the canonical map $(f, g) : Z \to X \times_W Y$ belongs to $\mathcal{L}$. 
GBMT ⇒ BMT

Remarks

- if \( f \) is \( m \)-connected, then \( \Delta(f) \) is \( (m - 1) \) connected;
- If \( f \) is \( m \)-connected and \( g \) is \( n \)-connected, then \( f \square g \) is \( (m + n + 2) \) connected.

Thus, if \( f \) is \( m \)-connected and \( g \) is \( n \)-connected, then \( \Delta(f) \square \Delta(g) \) is \( (m + n) \) connected.

Therefore: \( GBMT \Rightarrow BMT \).
The proof of the GBMT depends on the following descent lemma.

Lemma

[ABFJ] Let \((\mathcal{L}, \mathcal{R})\) be a modality in an \(\infty\)-topos \(\mathcal{E}\) and let

\[
\begin{array}{ccc}
f & \xrightarrow{\beta} & h \\
\downarrow{\alpha} & & \downarrow{\delta} \\
g & \xrightarrow{\gamma} & k
\end{array}
\]

be a pushout square in \(\mathcal{E}^{[1]}\). If the square \(\alpha\) and \(\beta\) are \(\mathcal{L}\)-cartesian, then so are the squares \(\delta\) and \(\gamma\).
Let $S$ be the quasicategory of spaces and let $S\dot{\;}$ be the quasicategory of \textit{pointed} spaces.

A functor $F : S\dot{\;} \to S$ is said to be a \textit{homotopy functor} if it preserves directed colimits.

Recall that Goodwillie’s Calculus associates to a homotopy functor $F : S\dot{\;} \to S$ a tower of approximations by \textit{n-excisive} functors:

$$P_0(F) \leftarrow P_1(F) \leftarrow P_2(F) \leftarrow \cdots .$$  \hspace{1cm} (5)

The first approximation $P_0(F)$ is the constant functor with value $F(0)$. 

There is an analogy between the Goodwillie tower of a functor $F$ and the Postnikov tower of a space $X$

$$S_0(X) \leftarrow S_1(X) \leftarrow S_2(X) \leftarrow \cdots.$$ \hspace{1cm} (6)

The first approximation $S_0(X)$ is the set $\pi_0(X)$ of connected components of $X$.

The classical Blakers-Massey theorem is a powerful tool for studying the Postnikov tower.

Biedermann’s question: Is there a Blakers-Massey theorem for Goodwillie’s Calculus?
Goodwillie’s toposes

A homotopy functor $F : S_\bullet \to S$ is entirely determined by its restriction to the sub-quasicategory of finite pointed spaces $Fin_\bullet \subset S_\bullet$.

It follows that the quasi-category of homotopy functors $S_\bullet \to S$ is equivalent to the quasi-category $[Fin_\bullet, S]$ of all functors $Fin_\bullet \to S$.

The functor $P_n : [Fin_\bullet, S] \to [Fin_\bullet, S]$ is a left exact reflection by a theorem of Goodwillie.

(Biedermann, Rezk) The quasicategory $[Fin_\bullet, S]$ is an $\infty$-topos, since it is a presheaf category. Hence the quasi-category $[Fin_\bullet, S]_n$ of $n$-excisive functors is an $\infty$-topos.
$P$-equivalences

Let $\mathcal{E}$ be an $\infty$-topos and $P : \mathcal{E} \to \mathcal{E}$ a left exact reflection.

A map $f : X \to Y$ in $\mathcal{E}$ is said to be a $P$-equivalence if the map $P(f) : P(X) \to P(Y)$ is an equivalence.

A map $f : X \to Y$ is said to be a $P$-local if the naturality square

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & P(X) \\
\downarrow f & & \downarrow P(f) \\
Y & \xrightarrow{\eta_Y} & P(Y)
\end{array}
$$

is cartesian.

This defines a modality $(\mathcal{L}_P, \mathcal{R}_P)$, where $\mathcal{L}_P$ is the class of $P$-equivalences and $\mathcal{R}_P$ is the class of $P$-local maps.

The modality $(\mathcal{L}_P, \mathcal{R}_P)$ is said to be left exact.
The analogy

<table>
<thead>
<tr>
<th>spaces</th>
<th>homotopy functors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Postnikov tower</td>
<td>Goodwillie Tower</td>
</tr>
<tr>
<td>$n$-connected maps</td>
<td>$P_n$-equivalences</td>
</tr>
<tr>
<td>$n$-truncated maps</td>
<td>$P_n$-local maps</td>
</tr>
<tr>
<td>BM theorem</td>
<td>BM theorem for GC</td>
</tr>
</tbody>
</table>
BM theorem for Goodwillie calculus

**Theorem**
[ABFJ] Let

$$
\begin{array}{ccc}
F & \xrightarrow{g} & H \\
\downarrow{f} & & \downarrow{} \\
G & \xrightarrow{} & K
\end{array}
$$

be a homotopy pushout square of homotopy functors. If $f$ is a $P_m$-equivalence and $g$ is a $P_n$-equivalence, then the induced map $(f, g) : Z \rightarrow G \times^h_K H$ in the homotopy pullback is a $P_{m+n+1}$-equivalence.

The proof uses the GBM theorem and the following lemma:

**Lemma**
[ABFJ] The pushout product $f \Box g$ of a $P_m$-equivalence $f$ with a $P_n$-equivalence $g$ is a $P_{m+n+1}$-equivalence.
Some applications

A homotopy functor $F$ is said to be $n$-homogenous if $P_n(F) = F$ and $P_{n-1}F = P_0F$. The space $P_0(F) = F(0)$ is the base of $F$.

**Theorem**
*Goodwillie* The category of $n$-homogenous homotopy functors over a fixed base is stable for $n \geq 1$.

A homotopy functor $F$ is said to be $n$-reduced if $P_n(F) = 0$.

**Theorem**
*Arone-Dwyer-Lesh* If a homotopy functor $F$ is $n$-reduced and $(2n - 1)$-excisive, then it is infinitely deloopable.
Some applications

Theorem

(Goodwillie) If $n \geq 1$ and $F$ is a $n$-excisive functor with base $A := F(0)$, then there exists a $n$-homogenous functor $BK$ with base $A$ and a pullback square

$$
\begin{array}{ccc}
F & \longrightarrow & A \\
\downarrow & & \downarrow \\
P_{n-1}F & \longrightarrow & K
\end{array}
$$

Remark: the map $A \rightarrow K$ is an effective epimorphism and $G := \Omega_A(K)$ is a group object over $A$. This shows that the map $F \rightarrow P_{n-1}F$ is a principal $G$-fibration and that the map $P_{n-1}F \rightarrow K$ is a $k$-invariant.
Vladimir Voevodsky was a visionary. I hope his dream of univalent foundation will become true! Thank you!
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