A Tight Bound for Hypergraph Regularity

Guy Moshkovitz

Harvard University

Joint work with Asaf Shapira
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\textbf{Theorem (Graph regularity lemma (informal), Szemerédi ’78)}

\textit{The vertex set of every graph can be partitioned into a bounded number of parts such that almost all the bipartite graphs induced by pairs of parts in the partition are $\epsilon$-quasirandom.}
The Graph Regularity Lemma

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![Diagram of a graph and its partition](image)
Early applications:

- Tight bound for Ramsey-Turán problem for $K_4$ [Szemerédi ’72]
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Graph Regularity Lemma – Applications

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- The number of $H$-free graphs [Erdős-Frankl-Rödl ’86]
The Asymptotic Number of Graphs not Containing a Fixed Subgraph and a Problem for Hypergraphs Having No Exponent

P. Erdős¹, P. Frankl² and V. Rödl³

¹ Mathematical Institute of the Hungarian Academy of Science, P.O.B. 127, 1364 Budapest, Hungary
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Problem 6.1. Suppose \( H \) is a \( K_{t}(l,r) \)-free \( r \)-uniform hypergraph on \( n \) vertices, \( t > r \). Let \( \varepsilon \) be an arbitrarily small positive real \( n > n_{0}(\varepsilon, r, t, l) \). Is it possible to remove \( r \) edges from \( H \) so that the remaining hypergraph is \( K_{t}(r) \)-free?
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**Remark added in proof.** Problem 6.1 has been recently positively answered by P. Frankl and V. Rödl. The proof uses an extension of Szemerédi’s regularity lemma to hypergraphs.
20 Years Later... The Hypergraph Regularity Lemma

The main difficulty
Which notion of regularity/quasirandomness to use?
Should: 1. hold for all hypergraphs & 2. have a counting lemma

Theorem (Triangle Counting Lemma)
If $G$ is an $n \times n \times n$ tripartite graph whose 3 bipartite graphs are $\epsilon$-regular of densities $\alpha, \beta, \gamma$ then the number of triangles in $G$ is $(\alpha \beta \gamma \pm 7 \epsilon) n^3$. 

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Bad Example

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Example

There is a 4-partite 3-graph which is $K_{4}^{(3)}$-free even though each of the 4 triples of vertex classes is $o(1)$-regular:

- Let $T$ be a balanced 4-partite random tournament (where the direction of each $xy$ is chosen independently and uniformly).
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- Thus, each $xyz$ forms an edge in $H$ with probability $1/4$, and each of the 4 triples of vertex classes of $H$ is $o(1)$-regular.
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- Thus, each $xyz$ forms an edge in $H$ with probability $1/4$, and each of the 4 triples of vertex classes of $H$ is $o(1)$-regular.
- It is easy to see that $H$ is $K_4^{(3)}$-free.
Different versions of hypergraph regularity were proved by:
  - Frankl-Rödl '02, Rödl-Skopenkova '04, Nagle-Rödl-Schacht '06
  - Gowers '07
  - Tao '06
  - Rödl-Schacht '07

Remark [Chung-Graham-Wilson '89]
In graphs, discrepancy, codegree, eigenvalues, ... are poly-equivalent.
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Upper Bounds for Hypergraph Regularity

Common to all known proofs of the $k$-graph regularity lemma – their bound grows like $\text{Ack}_k$, the level-$k$ Ackermann function:
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- $\text{Ack}_1(n) = 2^n$
- $\text{Ack}_2(n) = T(n) = 2 \cdot 2 \cdot \ldots \cdot 2 \brace n \text{ times}$
- $\text{Ack}_3(n) = W(n) = T(\cdots (T(1)) \cdots) \quad (n \text{ compositions})$
- $\text{Ack}_4(n) = \ldots$
Detour: Applications

Original motivation—a combinatorial proof of Szemerédi’s Theorem:

**Theorem (Szemerédi ’74)**
\[
\forall \delta > 0, \ k \in \mathbb{N} \ \exists N = N(\delta, k):
\forall A \subseteq [N], \text{ if } |A| \geq \delta N \text{ then } A \text{ contains a } k\text{-term AP.}
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\forall A \subseteq [N]^d, \text{if } |A| \geq \delta N^d \text{ then } A \supseteq a + cX \text{ for some } a \in \mathbb{Z}^d, c \in \mathbb{N}.
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▶ Original proof uses ergodic theory, relies on Axiom of Choice.
▶ Only proof giving bounds relies on the hypergraph regularity lemma.

Fact
Improving upper bound for hypergraph regularity from $\text{Ack}k$ to $\text{Ack}k_0$ ⇒ first primitive recursive bound for Multidimensional Szemerédi’s Theorem.

▶ Obtaining such bounds for van der Waerden’s and Szemerédi’s Theorems (two special cases) were open problems for many decades (until solved by Shelah [JAMS ’89] and Gowers [GAFA ’01] respectively).
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In fact, we prove this lower bound for a new notion of regularity which, compared to previous notions, is:

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Gowers’ LB “reverse engineers” this UB, showing that constructing the partition using a sequence of exponential refinements is unavoidable.

More precisely, Gowers constructs a graph $G$, using a sequence of exponential refinements $P_1, P_2, \ldots$ of $V(G)$, with the following property: If $Z$ “approximately” refines $P_i$ but not $P_{i+1}$ then $Z$ is not $\epsilon$-regular.

Henceforth, we only consider 3-graph regularity.

The wowzer-type UB’s come from constructing a regular partition in a sequence of steps, each applying the graph regularity lemma and thus increasing the partition size by a tower-type function.

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Lower Bounds for Hypergraph Regularity
Summary:
A Barrier to Proving Hypergraph Regularity Lower Bounds

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Prove a wowzer-type (i.e., $A_3$) lower bound.
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**Barrier**
Any wowzer-type LB must imply a tower-type LB for relaxed graph regularity.

All known graph LB proofs fail to work vs. relaxed graph regularity.
Sparse Regular Approximation Lemma (SRAL)

SRAL

Input: $G$ with $pn^2$ edges.
Freedom: add/remove $1\% \cdot pn^2$ edges.
Goal: find a (small) $p^{10}$-regular partition.

Trivial upper bound: $T(1/p^{50})$.

Lower bound: ?

All previous constructions were not resilient to a constant fraction of edge modification.

Intuition
They were iterative, constructing the graph in “layers”. However, if one is allowed to modify 1% of the edges, one can essentially stop the construction at a stage where the graph still has a regular partition of constant order.
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Theorem (LB for SRAL, M.-Shapira '17)

Lower bound: $T(\Omega(\log \frac{1}{p}))$. 

Remark

The same paper also proves a matching upper bound for SRAL, and deduces Fox's celebrated $T(\mathcal{O}(\log \frac{1}{\epsilon}))$ bound [Ann. of Math. '11] for the graph removal lemma.
Bounds for SRAL

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An Even Weaker Notion of Graph Regularity

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**Definition ((δ)-regularity for graphs)**

- A bipartite graph on \((A, B)\) is \((\delta)\)-regular:
  \[ \forall A' \subseteq A, B' \subseteq B, \text{ if } |A'| \geq \delta |A|, |B'| \geq \delta |B| \text{ then } d(A', B') \geq \frac{1}{2} d(A, B). \]
An Even Weaker Notion of Graph Regularity

It turns out SRAL lower bound is not weak enough. We define a notion which is at the “correct level of strength”:

**Definition (\(\langle \delta \rangle\)-regularity for graphs)**

- A bipartite graph on \((A, B)\) is \(\langle \delta \rangle\)-regular:
  \[\forall A' \subseteq A, B' \subseteq B, \text{ if } |A'| \geq \delta |A|, |B'| \geq \delta |B| \text{ then } d(A', B') \geq \frac{1}{2} d(A, B).\]

- \(\mathcal{P}\) is a \(\langle \delta \rangle\)-regular partition of \(G\):
  Can modify \(\leq \delta \cdot e(G)\) edges so \(\forall A \neq B \in \mathcal{P}, G'[A, B] \) is \(\langle \delta \rangle\)-regular.
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Important difference from \(\epsilon\)-regularity: Can prove LB for \(⟨2^{-30})⟩\)-regularity.
Our Lower Bounds, Formally

Theorem (LB for graph $\langle \delta \rangle$-regularity, M.-Shapira ’18+)

$\forall p \in (0, 1) \ \exists \text{graph } G \text{ of density } p :$

every $\langle 2^{-30} \rangle$-regular partition of $G$ is of order $\geq \Theta(\log \frac{1}{p})$. 

Next goal

Lift LB for graph $\langle \delta \rangle$-regularity to a LB for $3$-graph $\langle \delta \rangle$-regularity.

Theorem (Main result (for 3-graphs), M.-Shapira ’18+)

$\forall p \in (0, 1) \ \exists 3$-graph $H$ of density $p :$

every $\langle 2^{-73} \rangle$-regular partition of $H$ is of order $\geq \Theta(\log \frac{1}{p})$. 
Our Lower Bounds, Formally

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$\forall p \in (0, 1) \exists$ graph $G$ of density $p$:

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Lift LB for graph $\langle \delta \rangle$-regularity to a LB for 3-graph $\langle \delta \rangle$-regularity.
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**Theorem (LB for graph $\langle \delta \rangle$-regularity, M.-Shapira '18+)**

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The 3-graph regularity lemmas of Frankl-Rödl and of Gowers both have a wowzer-type lower bound.
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In fact, even trivial versions of these notions are stronger than our notion.
Question

Is $\langle \delta \rangle$-regularity strong enough for counting small sub-hypergraphs?
Detour: How Strong is Our Lower Bound?

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**Answer**

It is not even strong enough to count triangles in graphs!
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**Lemma**

There are arbitrary large tripartite graphs of density \( \approx \delta^5 \) whose every pair of classes span a \( \langle \delta \rangle \)-regular graph and yet are triangle free.
Reminder:

**Definition ($\langle \delta \rangle$-regularity for graphs)**

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  \forall A' \subseteq A, B' \subseteq B, \text{ if } |A'| \geq \delta |A|, |B'| \geq \delta |B| \text{ then } d(A', B') \geq \frac{1}{2} d(A, B).
  \]
How Strong is our Lower Bound - cont.

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**Proof sketch.**

- A random \(k \times k \times k\) tripartite graph of density \(p \approx δ^5\) with \(k \approx δ^{-7}\) is both \(⟨δ⟩\)-regular and has \(\approx δ^{-6}\) triangles (\(\ll pk^2\)).
How Strong is our Lower Bound - cont.

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- Remove each triangle and take a blow-up; \(⟨δ⟩\)-regularity is preserved.
Main Result: Proof Sketch
Let $\mathcal{P}_1 \succ \cdots \succ \mathcal{P}_s$ be equipartitions with $|\mathcal{P}_{i+1}| = 2^c|\mathcal{P}_i|$. 
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**Theorem (Gowers ’97)**

\[ \exists \text{ graph } G \text{ such that } \forall \epsilon \text{-regular partition } \mathcal{Z} \text{ of } G \text{ we have:} \]

\[ \forall i : \mathcal{Z} \prec_x \mathcal{P}_i \Rightarrow \mathcal{Z} \prec_{4x} \mathcal{P}_{i+1} \text{ for } x \geq \sqrt{\epsilon}. \]

The implication was improved to give a simpler proof of the bound $T(1/\epsilon^c)$. 

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Guy Moshkovitz (Harvard University)  
Tight Bounds for Regularity Lemmas
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**Theorem (M.-Shapira ’16)**

There exists a graph $G$ such that for every $\epsilon$-regular partition $\mathcal{Z}$ of $G$ we have:

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\forall i : \quad \mathcal{Z} \prec_x \mathcal{P}_i \Rightarrow \mathcal{Z} \prec_{x+8\epsilon} \mathcal{P}_{i+1}.
\]
Core Construction (special case)

Henceforth:

- $\mathbf{L}$ and $\mathbf{R}$ are vertex classes,
- $\mathcal{L}_1 \succ \cdots \succ \mathcal{L}_s$ and $\mathcal{R}_1 \succ \cdots \succ \mathcal{R}_s$ are $s$ equipartitions of $\mathbf{L}$ and $\mathbf{R}$, respectively, with $|\mathcal{L}_i| = 2^c|R_i|$. 

Theorem (Core construction, special case)

$\exists$ bipartite graph $G$ on $(\mathbf{L}, \mathbf{R})$ with $d(G) = 2 - s$ such that $\forall$ $2$-regular partition $\langle \mathbf{L}, \mathbf{R} \rangle$ of $G$ we have: $\forall i$: $\mathcal{R}_i \prec 2 - 9 \mathcal{R}_i \Rightarrow \mathcal{L}_i \prec 2 - 9 \mathcal{L}_i$.

Main differences compared to [Gowers '97]:

- The partitions' orders can grow arbitrarily fast
- ...and $s$ can be arbitrarily large, with $d(G)$ decreasing with it.
- The graph's property is one sided.
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### Theorem (Core construction, special case)

\[ \exists \text{ bipartite graph } G \text{ on } (\mathbf{L}, \mathbf{R}) \text{ with } d(G) = 2^{-s} \text{ such that } \]
\[ \forall \langle 2^{-28} \rangle \text{-regular partition } (\mathbf{L}, \mathbf{R}) \text{ of } G \text{ we have: } \]
\[ \forall i : \mathbf{R} \lesssim_{2^{-9}} \mathbf{R}_i \Rightarrow \mathbf{L} \lesssim_{2^{-9}} \mathbf{L}_i . \]
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Henceforth:

- \( L \) and \( R \) are vertex classes,
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\forall i : \quad R \prec_{2^{-g}} R_i \quad \Rightarrow \quad L \prec_{2^{-g}} L_i .
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Main differences compared to [Gowers '97]:

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To prove our graph $\langle \delta \rangle$-regularity lower bound from Core Construction, put 4 copies along a 4-cycle.

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\[ |L_i| = |R_{i+1}| \]
Core Construction (general case)

Theorem (Core Construction)

\[ \exists \text{ equipartitions } G_1 \succ \cdots \succ G_s \text{ of } L \times R \text{ with } |G_j| = 2^j \text{ such that } \forall G \in G_j \forall \langle 2^{-28} \rangle\text{-regular partition } (\mathcal{L}, \mathcal{R}) \text{ of } G \text{ we have:} \]

\[ \forall i \leq j : \mathcal{R} \prec_{2^{-9}} \mathcal{R}_i \implies \mathcal{L} \prec_{2^{-9}} \mathcal{L}_i. \]
Why is Core Construction One-Sided?

- In order to prove a wowzer-type LB we will apply Core Construction with partitions whose orders grow as a wowzer-type function.
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- Had Core Construction held without the one-sided assumption then one would have been able to prove wowzer-type LB for graph $\langle \delta \rangle$-regularity and thus also for Szemerédi’s regularity lemma.
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- In order to prove a wowzer-type LB we will apply Core Construction with partitions whose orders grow as a wowzer-type function.
- Had Core Construction held without the one-sided assumption then one would have been able to prove wowzer-type LB for graph $\langle \delta \rangle$-regularity and thus also for Szemerédi’s regularity lemma.
- In other words, if one wishes to have a construction that holds with arbitrarily fast growing orders, then one has to introduce one-sidedness.
The Plan for 3-Graphs

Perhaps the most surprising aspect of our proof is that in order to construct a 3-graph we also use Core Construction in a somewhat unexpected way:

$L$ will be a complete bipartite graph $V_1 \times V_2$ (and $R$ will be $V_3$).

The $L_i$'s will be partitions of $V_1 \times V_2$ themselves given by another application of Core Construction.

The second application of Core Construction will "multiply" $L_i$ and $R_i$ to give a 3-graph which is hard for $\langle \delta \rangle$-regularity.
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\[
\begin{align*}
H & \\
V_3 & d(H) = p \\
V_1 & \\
V_2 & \\
G_H & \\
V_3 & \\
V_1 \times V_2 &
\end{align*}
\]
The Definition of 3-Graph $\langle \delta \rangle$-Regularity

A 2-partition $\mathcal{P}$ consists of a vertex equipartition $V_1, \ldots, V_t$, and an edge equipartition $K[V_i, V_j] = G_{1}^{i,j} \cup \cdots \cup G_{\ell}^{i,j} \ (\forall i \neq j)$.
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For 3-regularity, $\mathcal{P}$ itself has to satisfy a condition.
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For 3-regularity, $\mathcal{P}$ itself has to satisfy a condition.

**Definition ($\langle \delta \rangle$-good partition)**

A 2-partition is $\langle \delta \rangle$-good if every bipartite graph $G_{\ell}^{i,j}$ is $\langle \delta \rangle$-regular.
Definition (The auxiliary graph $G_H$)

Let $H$ be a 3-partite 3-graph $H$ on $(V^1, V^2, V^3)$. Define a bipartite graph $G_H = G_H(V^1, V^2 \times V^3)$ on $(V^1, V^2 \times V^3)$ by

$$E(G_H) = \{ (v_1, (v_2, v_3)) \mid (v_1, v_2, v_3) \in E(H) \}.$$
3-graph $\langle \delta \rangle$-regularity

**Definition ($\langle \delta \rangle$-regularity for 3-graphs)**

Let $H$ be a 3-partite 3-graph on $(V^1, V^2, V^3)$, and let $P$ be a $\langle \delta \rangle$-good 2-partition on $\{V^1, V^2, V^3\}$. $P$ is a $\langle \delta \rangle$-regular partition of $H$ if:

1. $P[V^1] \cup P[V^2 \times V^3]$ is a $\langle \delta \rangle$-regular partition of $G_H(V^1, V^2 \times V^3)$. 

Guy Moshkovitz (Harvard University) Tight Bounds for Regularity Lemmas
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$G_H$
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1. $\mathcal{P}[V^1] \cup \mathcal{P}[V^2 \times V^3]$ is a ⟨δ⟩-regular partition of $G_H(V^1, V^2 \times V^3)$,
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3. $\mathcal{P}[V^3] \cup \mathcal{P}[V^1 \times V^2]$ is a ⟨δ⟩-regular partition of $G_H(V^3, V^1 \times V^2)$.
Our 3-graph Construction

\[
H
\]

\[
V_3^{T(1), T(2), \ldots, T(W(\log \frac{1}{p}))}
\]

\[
G_H
\]

\[
V_3
\]

\[
V_1 \times V_2
\]

Property:
\[
P[V_3] \prec 2^{-9} V_3 i \quad \text{and} \quad P[V_2] \prec 2^{-9} V_2 i \Rightarrow P[V_1] \prec 2^{-9} V_1 i + 1.
\]

Finally, take several copies of \(H\) along a (tight) 6-cycle.
1. Apply Core Construction with \((L, R) = (V^1, V^2)\).
Apply Core Construction with \((L, R) = (V^1, V^2)\).
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Our 3-graph Construction

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2. Apply Core Construction with \((L, R) = (V^1 \times V^2, V^3)\).
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Our 3-graph Construction

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Finally, take several copies of \(H\) along a (tight) 6-cycle.
Open Question

We now know that “$k$-graph SRAL” has an $\text{Ack}_k(\Omega(\log \frac{1}{p}))$ lower bound.

▶ Prove a matching upper bound.
We now know that “$k$-graph SRAL” has an $\text{Ack}_k(\Omega(\log \frac{1}{p}))$ lower bound.

- Prove a matching upper bound.
- Deduce an $\text{Ack}_k(\Omega(\log \frac{1}{\epsilon}))$ bound for the $k$-graph removal lemma, thus improving the current bound $\text{Ack}_k(\Omega(\text{poly}(\frac{1}{\epsilon})))$. 

Open Questions
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Open Question

Come up with a weaker notion than hypergraph regularity that has primitive recursive bounds and yet is useful.
Thank you!
The construction of our 3-graph

For $1 \leq j \leq 3$ fix canonical partitions $\mathcal{V}_1^j \succ \mathcal{V}_2^j \succ \cdots$ with $|\mathcal{V}_i^j| \approx T(i)$. 
The construction of our 3-graph

For $1 \leq j \leq 3$ fix canonical partitions $\mathcal{V}_1^j \succ \mathcal{V}_2^j \succ \cdots$ with $|\mathcal{V}_i^j| \approx T(i)$. Apply Key Lemma twice:

1. $(L, R) = (\mathcal{V}^1, \mathcal{V}^2)$, $(L_i, R_i) = (\mathcal{V}_{i+1}^1, \mathcal{V}_i^2)$ to get $G_1 \succ G_2 \succ \cdots$. 

Finally, take several copies of $H$ along a small design.
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Take any graph in the last edge partition to get a 3-graph $H$. 

Main claim

If $H$ is $⟨\delta⟩$-regular relative to $\mathcal{P}$ and

\[ \mathcal{P}[V_3] \preceq 2^{-9} V_3 \]

and

\[ \mathcal{P}[V_2] \preceq 2^{-9} V_2 \]

then

\[ \mathcal{P}[V_1] \preceq 4\sqrt{\delta} V_{i+1}^1 \]
The construction of our 3-graph

For $1 \leq j \leq 3$ fix canonical partitions $\mathcal{V}_j^1 \succ \mathcal{V}_j^2 \succ \cdots$ with $|\mathcal{V}_j^i| \approx T(i)$. Apply Key Lemma twice:

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The construction of our 3-graph

For $1 \leq j \leq 3$ fix canonical partitions $\mathcal{V}_j^1 \succ \mathcal{V}_j^2 \succ \cdots$ with $|\mathcal{V}_j^i| \approx T(i)$. Apply Key Lemma twice:

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Main claim

If $H$ is $\langle \delta \rangle$-regular relative to $\mathcal{P}$ and

if $\mathcal{P}[V^3] \prec_{2-9} V^3_i$ and $\mathcal{P}[V^2] \prec_{2-9} V^2_i$ then $\mathcal{P}[V^1] \prec \sqrt{\delta} V^1_{i+1}$. 

Guy Moshkovitz (Harvard University)  
Tight Bounds for Regularity Lemmas
Main claim

If $H$ is $\langle \delta \rangle$-regular relative to $P$ and

if $P[V^3] \ll_{2^{-9}} V_i^3$ and $P[V^2] \ll_{2^{-9}} V_i^2$ then $P[V^1] \ll \sqrt{4} \delta V_{i+1}^1$. 

Suppose $W(j) \leq i < W(j+1)$:
Main claim

If \( H \) is \( \langle \delta \rangle \)-regular relative to \( \mathcal{P} \) and

\[
\text{if } \mathcal{P}[\mathbb{V}^3] \prec_{2-9} \mathbb{V}_i^3 \text{ and } \mathcal{P}[\mathbb{V}^2] \prec_{2-9} \mathbb{V}_i^2 \text{ then } \mathcal{P}[\mathbb{V}^1] \prec_{4/\delta} \mathbb{V}_{i+1}.
\]

Suppose \( W(j) \leq i < W(j + 1) \):

![Diagram with sets and relations]
The construction uses the following graph operation:

**Modified blow-up of a bipartite graph** $G$:

- replace each vertex $x$ of $G$ by a set of $2^{Ω(|V(G)|)}$ new vertices $X$
- replace each edge $(u, v)$ with a bipartite graph on $(U, V)$ as follows: letting $U' \subseteq U$ be a random half, replace $(u, v)$ by $K(U', V)$. 

Starting from $K_1, 1$, iteratively apply modified blow-ups $\log_2 p$ times. Each application increases the number of vertices exponentially and halves the density, so the resulting graph has density $p$ and $T(Ω(\log_2 p))$ vertices.

Intuition: If $G$ has a "unique" regular partition, then so does its modified blow-up.
The construction uses the following graph operation:

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Lower Bounds Proof for Graph $\langle \delta \rangle$-Regularity

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**Intuition**

If $G$ has a “unique” regular partition then so does its modified blow-up.
Arguably most important application of the graph regularity lemma:
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**Theorem (Triangle Removal Lemma, Ruzsa-Szemerédi ‘76)**

*For every $n$-vertex graph,*

\[
\# \text{edge-disjoint triangles} \geq \epsilon n^2 \implies \# \text{triangles} \geq f(\epsilon)n^3.
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Arguably most important application of the graph regularity lemma:

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For every $n$-vertex graph,

$$\#\text{edge-disjoint triangles} \geq \epsilon n^2 \implies \#\text{triangles} \geq f(\epsilon)n^3.$$ 

Application:

**Theorem (Roth’s Theorem, ‘53)**

For every subset $A \subseteq [n] = \{1, 2, \ldots, n\},$

$$|A| \geq \epsilon n \text{ and } n \geq n_0(\epsilon) \implies A \text{ contains a 3-AP.}$$
Theorem (Roth’s Theorem)

∀ A ⊆ [n]: |A| ≥ 0.01n ⇒ A contains a 3-AP.

Proof.

 Observation: a pair of (ordered) APs cannot agree on two elements.
Theorem (Roth’s Theorem)

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- Observation: a pair of (ordered) APs cannot agree on two elements.
- Consider all (ordered) 3-APs (x, x + a, x + 2a) with x ∈ [n], a ∈ A.
RL \Rightarrow \text{Roth’s Theorem}

**Theorem (Roth’s Theorem)**

\[ \forall A \subseteq [n] : \quad |A| \geq 0.01n \Rightarrow A \text{ contains a 3-AP.} \]

**Proof.**

- Observation: a pair of (ordered) APs cannot agree on two elements.
- Consider all (ordered) 3-APs \((x, x + a, x + 2a)\) with \(x \in [n], a \in A\).
- Consider the corresponding tripartite graph (on \([n] \cup [2n] \cup [3n]\)).
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- #edge-disjoint-triangles is n|A| ≥ 0.01n². TRL ⇒ another triangle.
Theorem (Roth’s Theorem)

\[ \forall A \subseteq [n]: \quad |A| \geq 0.01n \Rightarrow A \text{ contains a 3-AP.} \]

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- \#edge-disjoint-triangles is \(n|A| \geq 0.01n^2\). TRL \(\Rightarrow\) another triangle.
- Its elements: \((y, y + \alpha, y + 2\alpha')\) with \(\alpha \neq \alpha' \in A\).
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  - We have \( (y + 2\alpha') - (y + \alpha) = 2\alpha' - \alpha \in A \).
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  - We have \((y + 2\alpha') - (y + \alpha) = 2\alpha' - \alpha \in A\).
  - We found a (non-trivial) 3-AP in \(A\): \((\alpha, \alpha', 2\alpha' - \alpha)\).
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- \#edge-disjoint-triangles is \(n|A| ≥ 0.01n^2\). TRL ⇒ another triangle.
- Its elements: \((y, y + α, y + 2α')\) with \(α ≠ α' \in A\).
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  - We found a (non-trivial) 3-AP in \(A\): \((α, α', 2α' - α)\).

Best known bounds:

\[
\varepsilon \ln(1/\varepsilon) ≤ \text{Rem}(\varepsilon) ≤ T(1/\varepsilon)
\]

\[
n^{-1/\sqrt{\log n}} ≤ r_3(n) ≤ \approx (\log n)^{-1}
\]