Geometric Complexity Theory via Algebraic Combinatorics

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IAS, CSDM Seminar
(Boolean) Complexity

**Input**: string of \( n \) bits, i.e. size\((input) = n\).

**Decision problems:**

Is there an object, s.t....?  

\[
\begin{align*}
\text{P} & = \text{solution can be found in time Poly}(n) \\
\text{NP} & = \text{solution can be verified in Poly}(n) \text{ (polynomial witness)} \\
\text{NP –Complete} & = \text{in NP}, \text{ and every NP problem can be reduced to it poly time; e.g.}
\end{align*}
\]

**Counting problems:**

Compute \( F(input) = ? \)

\[
\begin{align*}
\text{FP} & = \text{solution can be found in time Poly}(n) \\
\#\text{P} & = \text{NP counting analogue; informally – } F(input) \text{ counts Exp-many objects, whose verification is in P.}
\end{align*}
\]
(Boolean) Complexity

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Is there an object, s.t.... ?

- **P** = solution can be found in time $\text{Poly}(n)$
- **NP** = solution can be verified in $\text{Poly}(n)$ (polynomial witness)
- **NP-Complete** = in NP, and every NP problem can be reduced to it poly time;

**The P vs NP Problem:**

Is P = NP? Algebraic version: is VP = VNP?

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Compute $F(input)$ =?

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- **#P** = NP counting analogue; informally – $F(input)$ counts Exp-many objects, whose verification is in P.
(Boolean) Complexity

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$P$ = solution can be found in time Poly($n$)

$NP$ = solution can be verified in Poly($n$) (polynomial witness)

NP–Complete = in NP, and every NP problem can be reduced to it poly time;

The P vs NP Problem:

Is $P = NP$? Algebraic version: is $VP = VNP$?

An approach [Mulmuley, Sohoni]: Geometric Complexity Theory

Counting problems:

Compute $F(input) =$?

$FP$ = solution can be found in time Poly($n$)

#$P$ = NP counting analogue; informally – $F(input)$ counts Exp-many objects, whose verification is in $P$.
VP vs VNP: determinant vs permanent

**Arithmetic Circuits:**

\[ y = 3x_1 + x_1x_2 \]

Polynomials \( f_n \in \mathbb{F}[X_1, \ldots, X_n] \). Circuit – nodes are +, × gates, input – \( X_1, \ldots, X_n \) and constants from \( \mathbb{F} \).

**Class VP (Valliant’s P):**

polynomials that can be computed with \( \text{poly}(n) \) large circuit (size of the associated graph).

**Class VNP:**

the class of polynomials \( f_n \), s.t.

\[
\exists g_n \in \text{VP} \quad \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n) = f_n
\]
VP vs VNP: determinant vs permanent

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the class of polynomials \( f_n \), s.t. \( \exists g_n \in \text{VP} \) with
\[
    f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).
\]

**Theorem**[Bürgisser]: If VP = VNP, then P = NP if \( \mathbb{F} \) - finite or the Generalized Riemann Hypothesis holds.

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VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial \( p \) in any number of variables there exists some \( n \) such that

\[
p = \det(A),
\]

where \( A \) is an \( n \times n \) matrix whose entries are affine linear polynomials. The smallest \( n \) possible is called the determinantal complexity \( dc(p) \).

Example: \( p = x_1^2 + x_1x_2 + x_2x_3 + 2x_1 \), then

\[
p = \det \begin{bmatrix} x_1 + 2 & x_2 \\ -x_3 + 2 & x_1 + x_2 \end{bmatrix}, \quad dc(p) = 2
\]
VP vs VNP: determinant vs permanent

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$$p = \text{det}(A),$$

where $A$ is an $n \times n$ matrix whose entries are affine linear polynomials. The smallest $n$ possible is called the determinant complexity $\text{dc}(p)$.

The permanent:

$$\text{perm} := \sum_{\sigma \in S_m} \prod_{i=1}^{m} X_{i,\sigma(i)}.$$

Theorem: [Valiant] $\text{perm}$ is VNP-complete.

Conjecture (Valiant, $\text{VP} \neq \text{VNP}$ equivalent)

$\text{dc}(\text{perm})$ grows superpolynomially.
VP vs VNP: determinant vs permanent

**Universality of the determinant [Cohn, Valiant]:**
For every polynomial $p$ in any number of variables there exists some $n$ such that

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**The permanent:**

$$\text{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} X_{i,\sigma(i)}.$$

**Theorem:** [Valiant] $\text{per}_m$ is VNP-complete.

**Conjecture (Valiant, VP $\neq$ VNP equivalent)**

$dc(\text{per}_m)$ grows superpolynomially.

**Known:** $dc(\text{per}_m) \leq 2^m - 1$ (Grenet 2011), $dc(\text{per}_m) \geq \frac{m^2}{2}$ (Mignon, Ressayre, 2004). Ryser’s formula:

$$\text{per}_m(X) = (-1)^m \sum_{S \subseteq [1..m]} (-1)^{|S|} \prod_{i=1}^{m} (\sum_{j \in S} X_{i,j})$$

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**Geometric Complexity Theory**

$GL_N$ action on polynomials: $A \in GL_N(\mathbb{C})$, $v := (X_1, \ldots, X_N)$, $f \in \mathbb{C}[X_1, \ldots, X_N]$, then $A.f = f(A^{-1}v)$
(replaces variables with linear forms)

$GL_{n^2} \det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – **determinant orbit**.

$\Omega_n := GL_{n^2} \det_n$ - **determinant orbit closure**.

$\text{per}_m^n := (X_{1,1})^{n-m}\text{per}_m$ – **the padded permanent**.
Geometric Complexity Theory

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**Proposition ( Lower bounds via geometry )**

If $\per^n_m \notin \overline{GL_n^2 \det_n}$, then $dc(\per^m) > n.$
Geometric Complexity Theory

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**Proposition (Lower bounds via geometry)**

If $\text{per}_m \notin GL_{n^2} \det_n$, then $dc(\text{per}_m) > n$.

**Conjecture (GCT: Mulmuley and Sohoni)**

$max\{ n : \text{per}_m \notin \overline{GL_{n^2} \det_n} \} (\leq dc(\text{per}_m))$ grows superpolynomially.

$\text{per}_m \in \overline{GL_{n^2} \det_n} \iff \overline{GL_{n^2} \text{per}_m} \subseteq \overline{GL_{n^2} \det_n}$. $\Gamma^\prime_m := \overline{\text{per}_m} \subseteq \Omega_n$
Geometric Complexity Theory

Proposition (Lower bounds via geometry)

If $\text{per}_m^n \notin GL_{n^2} \text{det}_n$, then $dc(\text{per}_m) > n$.

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$\text{per}_m^n \in GL_{n^2} \text{det}_n \iff \underbrace{GL_{n^2} \text{per}_m^n}_{=: \Gamma_m} \subseteq \underbrace{GL_{n^2} \text{det}_n}_{\Omega_n}$. 
Geometric Complexity Theory

Proposition (Lower bounds via geometry)
If \( \text{per}_m^n \not\in \overline{GL_n^2 \det_n} \), then \( \text{dc}(\text{per}_m) > n \).

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\[
\max \{ n : \text{per}_m^n \not\in \overline{GL_n^2 \det_n} \} (\leq \text{dc}(\text{per}_m)) \text{ grows superpolynomially.}
\]

\[ \text{per}_m^n \in \overline{GL_n^2 \det_n} \iff \overline{GL_n^2 \text{per}_m^n} \subseteq \overline{GL_n^2 \det_n}. \]

Exploit the symmetry! Coordinate rings as \( GL_n^2 \) representations:
\[
\mathbb{C}[\overline{GL_n^2 \det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[\overline{GL_n^2 \text{per}_m^n}]_d \simeq \bigoplus_{\lambda} V^{\oplus \gamma_{\lambda,d,n,m}},
\]

Definition (Representation theoretic obstruction)
If \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \), then \( \lambda \) is a representation theoretic obstruction.
Its existence shows \( \overline{GL_n^2 \text{per}_m^n} \not\subseteq \overline{GL_n^2 \det_n} \) and so \( \text{dc}(\text{per}_m) > n \).
(Non)existence of obstructions

\[ \mathbb{C}[GL_n^2 \text{det}_n]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_n^2 \text{per}_m]_d \cong \bigoplus_{\lambda} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

If \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \), then \( \lambda \) is a representation theoretic obstruction and \( d c(\text{per}_m) > n \). If \( n > \text{poly}(m) \implies \text{VP} \neq \text{VNP} \).
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2} \text{det}_n]_d \cong \bigoplus_{\lambda \vdash nd} V_\lambda^{\delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2} \text{per}_m]_d \cong \bigoplus_{\lambda} V_\lambda^{\gamma_{\lambda,d,n,m}}, \]

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Conjecture (GCT: Mulmuley-Sohoni)

There exist representation theoretic obstructions that show superpolynomial lower bounds on \( dc(\text{per}_m) \).
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**Conjecture (GCT: Mulmuley-Sohoni)**

There exist representation theoretic obstructions that show superpolynomial lower bounds on \( dc(\text{per}_m) \).

If also \( \delta_{\lambda,d,n} = 0 \), then \( \lambda \) is an occurrence obstruction.

**Conjecture (Mulmuley and Sohoni)**

There exist occurrence obstructions that show superpolynomial lower bounds on \( dc(\text{per}_m) \).
(Non)existence of obstructions

\[ \mathbb{C}[GL_n^2\mathrm{det}_n]_d \simeq \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_n^2\mathrm{per}_m]_d \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

If \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \), then \( \lambda \) is a representation theoretic obstruction and \( dc(\mathrm{per}_m) > n \). If \( n > poly(m) \) \( \implies \) \( VP \neq VNP \).

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Conjecture (Mulmuley and Sohoni)

There exist occurrence obstructions that show superpolynomial lower bounds on \( dc(\mathrm{per}_m) \).

Theorem (Bürgisser-Ikenmeyer-P(FOCS 2016))

This Conjecture is false.
(Non)existence of obstructions

\[ \mathbb{C}[GL_{n^2}\det_n]_d \cong \bigoplus_{\lambda \vdash nd} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \mathbb{C}[GL_{n^2}\per_m]_d \cong \bigoplus_{\lambda} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}, \]

If \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \), then \( \lambda \) is a representation theoretic obstruction and \( dc(\per_m) > n \). If \( n > poly(m) \implies VP \neq VNP \).

**Question:** What are these \( \delta_{\lambda,d,n} \) and \( \gamma_{\lambda,d,n,m} \)??
(Non)existence of obstructions

\[ C[GL_n^2 \text{det}]_d \simeq \bigoplus_{\lambda \vdash nd} V^\lambda \otimes \delta_{\lambda,d,n}, \quad C[GL_n^2 \text{per}_m]_d \simeq \bigoplus_{\lambda} V^\lambda \otimes \gamma_{\lambda,d,n,m}, \]

If \( \delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m} \), then \( \lambda \) is a representation theoretic obstruction and \( dc(\text{per}_m) > n \). If \( n > \text{poly}(m) \implies \text{VP} \neq \text{VNP} \).

**Question:** What are these \( \delta_{\lambda,d,n} \) and \( \gamma_{\lambda,d,n,m} \)?

**Kronecker coefficients** of the Symmetric Group:

\[ \delta_{\lambda,d,n} \leq sk(\lambda,n^d) \leq g(\lambda,n^d,n^d) \]

(Symmetric Kronecker:

\( sk(\lambda,\mu) := \dim \text{Hom}_{S|_{\lambda}}(S^\lambda, S^2(S^\mu)) = \text{mult}_\lambda C[GL_n^2 \text{det}]_d \)

**Plethysm coefficients:** of \( GL \).

\[ a_\lambda(d[n]) := \text{mult}_\lambda \text{Sym}^d(\text{Sym}^n(V)) \geq \gamma_{\lambda,d,n,m}. \]

**Problem (GCT program, “easy version”)**

Find \( \lambda \), such that the \( sk(\lambda,(n^d)) < a_\lambda(d[n]) \)?
Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many $m$, there exists a partition $\lambda$ occurring in $\mathbb{C}[GL_n^2 X_{11}^{n-m} \text{per}_m]$ but not in $\mathbb{C}[GL_n^2 \cdot \text{det}_n]$, where $n = m^c$. 
Positivity towards negativity

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Theorem (Ikenmeyer-P (2015, FOCS’16))
Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$ (so $\text{mult}_\lambda\mathbb{C}[GL_{n^2} \det_n] = 0$), then $\text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}\text{per}_m]) = 0$.
Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)
For all \( c \in \mathbb{N}_{\geq 1} \), for infinitely many \( m \), there exists a partition \( \lambda \) occurring in \( \mathbb{C}[GL_{n^2}X_{11}^{n-m}\per_m] \) but not in \( \mathbb{C}[GL_{n^2}\cdot \det_n] \), where \( n = m^c \).

Theorem (Ikenmeyer-P (2015, FOCS’16))
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Theorem (Bürgisser-Ikenmeyer-P (FOCS’16))
Let \( n, d, m \) be positive integers with \( n \geq m^{25} \) and \( \lambda \vdash nd \). If \( \lambda \) occurs in \( \mathbb{C}[GL_{n^2}X_{11}^{n-m}\per_m] \), then \( \lambda \) also occurs in \( \mathbb{C}[GL_{n^2}\cdot \det_n] \). In particular, the Conjecture is false, there are no “occurrence obstructions”.

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Classical problems in Algebraic Combinatorics

Irreducible representations of the symmetric group $S_n$:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the Specht modules $S_\lambda$
Classical problems in Algebraic Combinatorics

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( group homomorphisms $S_n \to GL_N(\mathbb{C})$ )

are the Specht modules $S_\lambda$, indexed by integer partitions $\lambda \vdash n$:

\[ \lambda = (\lambda_1, \ldots, \lambda_\ell), \]
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0, \]
\[ \lambda_1 + \lambda_2 + \cdots = n, \text{ length } \ell(\lambda) = \ell \text{ (= number of nonzero parts)} \]

**Young diagram** of $\lambda$:

```
  1 2 3
 4 5  
 3 5  
```

( $\lambda = (5, 3, 2)$, $\ell(\lambda) = 3$, $n = |\lambda| = 5 + 3 + 2 = 10$).

**Basis for $S_\lambda$: Standard Young Tableaux** of shape $\lambda$:

```
1 2 3  | 1 2 4  | 1 2 5  | 1 3 4  | 1 3 5  
4 5   | 3 5   | 3 4   | 2 5   | 2 4   
```
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are the **Specht modules** $S_\lambda$

Tensor product decomposition:

$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} (\ldots) S_\nu$$
Classical problems in Algebraic Combinatorics

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Tensor product decomposition:

$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu) S_\nu$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$
Classical problems in Algebraic Combinatorics

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$S_n$ → $GL_N(\mathbb{C})$

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Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $S_\nu$ in $S_\lambda \otimes S_\mu$

$g(\lambda, \mu, \nu) = \dim \text{Hom}_{S_n}(S_\nu, S_\lambda \otimes S_\mu)$

In terms of $GL(\mathbb{C}^m)$ modules $V_\lambda, V_\mu, V_\nu$

$\text{Sym}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m) = \bigoplus_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) V_\lambda \otimes V_\mu \otimes V_\nu$
A bit of history

1873: Lie groups, *Lie, Klein*....

1896: Representations of finite groups, *Frobenius* ...

1923: Representations of Lie groups, *H. Weyl*. Quantum mechanics, *von Neumann*


$$V_\lambda \otimes V_\mu = \bigoplus_\nu c^\nu_{\lambda \mu} V_\nu$$

$c^\nu_{\lambda \mu}$ – Littlewood-Richardson coefficients.
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1934: Tensor products of irreducible representations of Lie groups: 

\[ V_\lambda \otimes V_\mu = \bigoplus \nu c^{\nu}_{\lambda \mu} V_\nu \]

\( c^{\nu}_{\lambda \mu} \) – Littlewood-Richardson coefficients.

**Theorem (Littlewood-Richardson, 1934)**

*The coefficient \( c^{\nu}_{\lambda \mu} \) is equal to the number of LR tableaux of shape \( \nu/\mu \) and type \( \lambda \).*
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The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.

$$\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
1 & 3 & 3 \\
\end{array}$$

(LR tableaux of shape $(7, 4, 3)/(3, 1)$ and type $(4, 3, 2)$. $c_{(3,1)(4,3,2)}^{(7,4,3)} = 2$)
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**Theorem (Littlewood-Richardson, 1934)**

*The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of LR tableaux of shape $\nu/\mu$ and type $\lambda$.*


$$S_\lambda \otimes S_\mu = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu) S_\nu$$
The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

*Find a positive combinatorial interpretation for* \( g(\lambda, \mu, \nu) \), *i.e. a family of combinatorial objects* \( O_{\lambda, \mu, \nu} \), *s.t. \( g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu} \).*
The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $O_{\lambda, \mu, \nu}$, s.t. $g(\lambda, \mu, \nu) = \#O_{\lambda, \mu, \nu}$. Alternatively, show that KRON is in $\#P$.

Classical motivation: (Littlewood–Richardson: for $c_{\lambda, \mu}^\nu$, $O_{\lambda, \mu, \nu} = \{ \text{LR tableaux of shape } \nu/\mu, \text{ type } \lambda \}$)

Theorem (Murnaghan)

If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c_{\lambda, \mu}^\nu.$$

Modern motivation:

1. A positive combinatorial formula "$\iff$" Computing Kronecker coefficients is in $\#P$.
2. Geometric Complexity Theory.
3. Invariant Theory, moment polytopes [see Bürgisser, Christandl, Mulmuley, Walter, Oliveira, Garg, Wigerson etc]
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Results since then:

Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- $\mu$ and $\nu$ are hooks, [Remmel, 1989]
- $\nu = (n-k, k)$ and $\lambda_1 \geq 2k - 1$, [Ballantine–Orellana, 2006]
- $\nu = (n-k, k), \lambda = (n-r, r)$ [Remmel–Whitehead, 1994; Blasiak–Mulmuley–Sohoni, 2013]
- $\nu = (n-k, 1^k)$, [Blasiak, 2012]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].
The combinatorics questions

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $O_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \# O_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in #P.

Bounds and positivity:

[Pak-P]: $g(\lambda, \mu, \mu) \geq |\chi^\lambda (2\mu_1 - 1, 2\mu_2 - 3, \ldots)$ when $\mu = \mu^T$.

Corollaries: $g(\lambda, \mu, \mu) > c \frac{2^{\sqrt{2k}}}{k^{9/4}}$ for $\lambda = (|\mu| - k, k)$, and $\text{diag}(\mu) \geq \sqrt{k}$.

Complexity results:

[Bürgisser-Ikenmeyer]: KRON is in GapP.
( Littlewood-Richardson, i.e. KRON’s special case, is #P-complete )

[Pak-P]: If $\nu$ is a hook, then KronPositivity is in P. If $\lambda, \mu, \nu$ have fixed length there exists a linear time algorithm for deciding $g(\lambda, \mu, \nu) > 0$.

[Ikenmeyer-Mulmuley-Walter]: KronPositivity is NP-hard.

[Bürgisser-Christandl-Mulmuley-Walter]: membership in the moment polytope is NP and coNP.
Back to GCT: Positivity towards negativity

Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many $m$, there exists a partition $\lambda$ occurring in $\mathbb{C}[GL_n^2 X_{11}^{n-m} \per_m]$ but not in $\mathbb{C}[GL_n^2 \cdot \text{det}_n]$, where $n = m^c$. 
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Theorem (Ikenmeyer-P (2015, FOCS’16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$ (so $\text{mult}_\lambda \mathbb{C}[GL_n^2 \det_n] = 0$), then $\text{mult}_\lambda (\mathbb{C}[GL_n^2 (X_{1,1}^{n-m} \text{per}_{m})]) = 0$. 
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Theorem (Bürgisser-Ikenmeyer-P (FOCS’16))
Let \( n, d, m \) be positive integers with \( n \geq m^{25} \) and \( \lambda \vdash nd \). If \( \lambda \) occurs in \( \mathbb{C}[GL_nX_{11}^{n-m} \text{per}_m] \), then \( \lambda \) also occurs in \( \mathbb{C}[GL_n \cdot \det_n] \). In particular, the Conjecture is false, there are no “occurrence obstructions”.
No occurrence obstructions I: positive Kroneckers

Theorem (Ikenmeyer-P (2015, FOCS’16))

Let \( n > 3m^4 \), \( \lambda \vdash nd \). If \( g(\lambda, n \times d, n \times d) = 0 \), then
\[
mult_{\lambda}(\mathbb{C}[GL_n^2(X_{1,1})^{n-m} \text{per}_m]) = 0.
\]

Proof:
\[
\bar{\lambda} := (\lambda_2, \lambda_3, \ldots) \vdash |\lambda| - \lambda_1
\]

Theorem (Kadish-Landsberg)

If \( \mult_{\lambda} \mathbb{C}[GL_n^2 X_{11}^{n-m} \text{per}_m] > 0 \), then \( |\bar{\lambda}| \leq md \) and \( \ell(\lambda) \leq m^2 \).

Theorem (Degree lower bound, [IP])

If \( |\bar{\lambda}| \leq md \) with \( a_{\lambda}(d[n]) > g(\lambda, n \times d, n \times d) \), then \( d > \frac{n}{m} \).
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Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n \times d, n \times d) = 0$, then

$$\mult_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}\per_m]) = 0.$$ 

Proof:

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If $|\bar{\lambda}| \leq md$ with $a_\lambda(d[n]) > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.

Theorem (Kronecker positivity, [IP] )

Let $\lambda \vdash dn$. Let $\mathcal{X} := \{(1), (2 \times 1), (4 \times 1), (6 \times 1), (2, 1), (3, 1)\}$.

(a) If $\bar{\lambda} \in \mathcal{X}$, then $a_\lambda(d[n]) = 0$.

(b) If $\bar{\lambda} \notin \mathcal{X}$ and $m \geq 3$ such that $\ell(\lambda) \leq m^2$, $|\bar{\lambda}| \leq md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$. 
Kronecker positivity I: hook-like $\lambda$s

Proposition (Ikenmeyer-P)

If there is an $a$, such that $g(\nu^k(a^2), a \times a, a \times a) > 0$ for all $k$, s.t. $k \not\in H^1(\rho)$ and $a^2 - k \not\in H^2(\rho)$ for some sets $H^1(\rho), H^2(\rho) \subset [\ell, 2a + 1]$, then $g(\nu^k(b^2), b \times b, b \times b) > 0$ for all $k$, s.t. $k \not\in H^1(\rho)$ and $b^2 - k \not\in H^2(\rho)$ for all $b \geq a$.

Proof idea:
Kronecker symmetries and semigroup properties:
Let $P_c = \{k : g(\nu^k(c^2), c \times c, c \times c) > 0\}$, we have

Claim: Suppose that $k \in P_c$, then $k, k + 2c + 1 \in P_{c+1}$.
Kronecker positivity I: hook-like $\lambda$s

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Corollary

We have that $g(\lambda, h \times w, h \times w) > 0$ for $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for most “small” partitions $\rho$ and all but finitely many values of $j$. 
Kronecker positivity II: squares, and decompositions

Theorem (Ikenmeyer-P)

Let $\nu \notin \mathcal{X}$ and $\ell = \max(\ell(\nu) + 1, 9)$, $a > 3\ell^{3/2}$, $b \geq 3\ell^2$ and $|\nu| \leq ab/6$. Then $g(\nu(ab), a \times b, a \times b) > 0$.

Proof sketch: decomposition + regrouping

$$\nu = \rho + \xi + \sum_{k=2}^{\ell} x_k((k-1) \times k) + \sum_{k=2}^{\ell} y_k((k-1) \times 2).$$
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Crucial facts:

- $g(k \times k, k \times k, k \times k) > 0$ [Bessenrodt-Behns].
- Transpositions: $g(\alpha, \beta, \gamma) = g(\alpha, \beta^T, \gamma^T)$ (with $\beta = \gamma = wxh$)
- Hooks and exceptional cases: $g(\lambda, h \times w, h \times w) > 0$ for all $\lambda = (hw - j - |\rho|, 1^j + \rho)$ for $|\rho| \leq 6$ and almost all $j$s.
- Semigroup property for positive triples: $g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max(g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)$.
Kronecker vs plethysm: inequality of multiplicities

**Stability** [Manivel]: \( g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d), \text{ as } n \to \infty. \)

\( St^1(\rho) := \{(n, d) | g((nd - |\rho|, \rho), n \times d, n \times d)\} = a_\rho(d). \)
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**Proposition (Ikenmeyer-P)**

*Fix \( \rho, \) and let \( (n, d) \in St^1(\rho), \) which is true in particular if \( n \geq |\rho|. \) Let \( \lambda = (nd-|\rho|, \rho). \) Then \( g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n]). \)*
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Fix $\rho$, and let $(n, d) \in St^1(\rho)$, which is true in particular if $n \geq |\rho|$. Let $\lambda = (nd - |\rho|, \rho)$. Then $g(\lambda, n \times d, n \times d) \geq a_{\lambda}(d[n])$.

**Proof:** $\lambda = \mu + d(n - m)$. Suppose $g(\lambda, n \times d, n \times d) < a_{\lambda}(d[n])$:

**KL’14:** If $\mu \vdash md$ then $mult_{\mu+d(n-m)}(\mathbb{C}[GL_n^{2}(X_{1,1})^{n-m}V_m]) \geq a_{\mu}(d[m])$, where $V_m := Sym^m \mathbb{C}^{m^2}$. 
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**Stability:** $g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d)$. 

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Kronecker vs plethysm: inequality of multiplicities

**Stability** [Manivel]: \( g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d) \), as \( n \to \infty \).
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**KL’14:** If \( \mu \vdash m d \) then \( \operatorname{mult}_{\mu + d(n - m)}(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq a_\mu(d[m]) \),
where \( V_m := \text{Sym}^m \mathbb{C}^{m^2} \).

**Stability:** \( g(\lambda, n \times d, n \times d) = g(\mu, m \times d, m \times d) \).

**GCT:** If \( \operatorname{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq g(\lambda, n \times d, n \times d) \) then \( dc(f_m) > n \) for some \( f_m \in V_m \).
Kronecker vs plethysm: inequality of multiplicities

Stability [Manivel]: \( g((nd - |\rho|, \rho), n \times d, n \times d) = a_\rho(d), \) as \( n \to \infty. \)

\[ \text{St}^1(\rho) \coloneqq \{(n, d) \mid g((nd - |\rho|, \rho), n \times d, n \times d) \} = a_\rho(d). \]

Proposition (Ikenmeyer-P)

Fix \( \rho, \) and let \((n, d) \in \text{St}^1(\rho), \) which is true in particular if \( n \geq |\rho|. \) Let \( \lambda = (nd - |\rho|, \rho). \) Then \( g(\lambda, n \times d, n \times d) \geq a_\lambda(d[n]). \)

Proof: \( \lambda = \mu + d(n - m). \) Suppose \( g(\lambda, n \times d, n \times d) < a_\lambda(d[n]): \)

\( \text{KL'14: If } \mu \vdash md \text{ then } \text{mult}_{\mu + d(n - m)}(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq a_\mu(d[m]), \)

where \( V_m \coloneqq \text{Sym}^m \mathbb{C}^{m^2}. \)

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GCT: If \( \text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq g(\lambda, n \times d, n \times d) \) then \( dc(f_m) > n \) for some \( f_m \in V_m. \)

\[ \implies \text{mult}_\lambda(\mathbb{C}[GL_{n^2}(X_{1,1})^{n-m}V_m]) \geq a_\mu(d[m]) = a_\lambda(d[n]) > g(\lambda, n \times d, n \times d) \]

\[ \implies \max_{f \in V_m} dc(f_m) > n \to \infty \]
Plethysm positivity

Theorem (Bürgisser-Ikenmeyer-P (FOSC’16))

Let $n, d, m$ be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If $\lambda$ occurs in $\mathbb{C}[GL_{n^2}X_{11}^{n-m}\text{per}_m]$, then $\lambda$ also occurs in $\mathbb{C}[GL_{n^2} \cdot \det_n]$. In particular, the Obstruction Existence Conjecture is false, there are no “occurrence obstructions”.

Proof ideas:
- For $\text{mult}_\lambda \mathbb{C}[GL_{n^2}X_{11}^{n-m}\text{per}_m] > 0$ we must have $\lambda_1 > d(n - m)$.
- (Valiant): $dc(X_1^s + \cdots + X_k^s) \leq ks$, hence...
  $\ell^{n-s}(v_1^s + \cdots + v_k^s) \in \Omega_n$ for $n \geq ks$.
- If a highest weight vector of weight $-\lambda$ does not vanish on $\Omega_n$ (or in particular, on the power sums), then $\delta_{\lambda,n} = \text{mult}_\lambda \mathbb{C}[\Omega_n] > 0$.
- Then $\delta_{\lambda,n} > 0$, because of the existence of $\lambda$-highest weight vectors in $\text{Sym}^d \text{Sym}^n V$, i.e. $a_\lambda(d[n]) > 0$ via explicit tableaux constructions:
  tableaux $T$ of shape $\lambda$, content $d \times n$,...

```
1 1 1 1 2 2 2 3 3 4 4 4 4 5
2 2 3 3 3
4 5 5 5 5
```
- decomposition into building blocks + regrouping
Next time:

- Matrix Powering vs permanent and the symmetric Kronecker coefficients.
- Iterated Matrix Multiplication vs permanent model.
- Matrix Multiplication lower bounds via GCT.
- Some combinatorics and bounds on the Kronecker coefficients.
Thank you!

**Geometric Complexity Theory**

The Kronecker coefficients of $S_n$

**Positivity**

Other models

Combinatorial primer: partitions

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**Algebraic Geometry**

$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

**Representation Theory**

**Complexity Theory**

$P \text{ vs } NP$

**Algebraic Combinatorics**

\[
s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 3 & 3 \\
2 & 3 & 3 \\
\end{array}
\]

**Statistical Mechanics**

**Probability**