Growth of Sobolev norms for the cubic NLS near 1D quasi-periodic solutions

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The defocusing cubic NLS

- Consider the equation

$$-iu_t + \Delta u = |u|^2 u$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

- NLS defines a complete flow in $H^s(\mathbb{T}^2)$, $s \geq 1$.

- Conserved quantities: the Hamiltonian and the mass

$$E[u] = \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \right) \frac{dx}{(2\pi)^2} dx$$
$$\mathcal{M}[u] = \int_{\mathbb{T}^2} |u|^2 dx.$$

Two very related problems

- Transfer of energy (weak turbulence).

- Lyapunov instability of invariant objects of the dynamical system.

Transfer of energy

- Fourier series of u,

$$u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{inx}.$$

- Can we have transfer of energy to high modes as $t \to +\infty$?
- We measure it with the growth of s-Sobolev norms ($s > 1$)

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s} |a_n(t)|^2\right)^{1/2},$$

  where $\langle n\rangle = (1+|n|^2)^{1/2}$.
- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \quad \text{for all} \quad t \geq 0.$$

- Simultaneous forward and backward cascade.

The I-team result

$$-iu_t + \Delta u = |u|^2 u$$

- Kuksin (1997): growth of Sobolev norms starting from an already large initial data.

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \delta, \qquad \|u(T)\|_{H^s} \geq K.$$

- The result also applies to $s \in (0,1)$.

More results

- M. G. and V. Kaloshin: $T \sim e^{\left(\frac{K}{\delta}\right)^{\beta}}$ for some $\beta > 1$.
- E. Haus and M. Procesi generalized the I-team result to the quintic NLS
$$-iu_t + \Delta u = |u|^4 u$$
- M. G, E. Haus and M. Procesi generalization to any other odd power.
- Bourgain conjecture: Take $s > 1$. There exists a solution such that
$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as} \qquad t \to +\infty.$$
- Z. Hani, B. Pausader, N. Tzvetkov, N. Visciglia proved unbounded growth for the cubic NLS in $\mathbb{R} \times \mathbb{T}^2$.
- It remains open in all compact manifolds.

Dynamical systems point of view

$$-iu_t + \Delta u = |u|^2 u$$

- $u = 0$ is an elliptic critical point: linearly stable in any H^s topology
- It is Lyapunov stable or unstable? It depends on the topology.
- Stability in L^2 and H^1 topology.
- I-team result $\equiv$ Instability in the H^s topology.

Instability of others invariant objects?

- What about the stability/instability of other invariant objects of the cubic NLS?

- For which time ranges?

- Can we attain Sobolev norm explosion starting arbitrarily close to an invariant object?

Hani approach towards the proof of Bourgain conjecture

- Bourgain conjecture: There exists a solution such that

$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as} \qquad t \to +\infty.$$

- Long-time strong instability: Let $(X, \|\cdot\|)$ be a Banach space and Φ^t a flow on it. We say Φ^t exhibits long-time strong instability near $u \in X$ if for every $\delta \in (0,1)$ and $K > 1$, there exists $u^* \in B_\delta(u)$ such that $\sup_{t>0} |\Phi^t(u^*)| > K$.
- Assume there exists $\mathcal{D} \subset \mathcal{F} \subset X$ with
    - $\mathcal{F}$ closed in X
    - $\mathcal{D}$ dense in $\mathcal{F}$.
    - All u in $\mathcal{D}$ has long-time strong instability within $\mathcal{F}$
- Then, there exists $u^* \in F$ such that $\|u^*(t)\|_{H^s} \to \infty$ as $t \to +\infty$.

Instability of plane waves

- Plane waves (periodic orbits): $u(t,x) = Ae^{i(mx-\omega t)}$ with $\omega = m^2 + A^2$.
- Stability result (Faou, Gauckler and Lubich 2013): Fix $N > 1$. There exists $s_0 > 0$ such that "many" plane waves are stable in H^s with $s \geq s_0$ and times $t \sim \delta^{-N}$ and arbitrary N where $\delta = \|u_0(x) - Ae^{imx}\|_{H^s}$.
- Many: For any m and a full measure set of A
- Instability result (Hani 2011): Fix $s \in (0,1)$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0) - Ae^{imx}\|_{H^s} \leq \delta, \qquad \|u(T)\|_{H^s} \geq K.$$

- Remark: Instabilities are only proven for $s \in (0,1)$.

Transfer of energy close to invariant tori

- Goal: Transfer of energy close to invariant (quasiperiodic) tori.
- We look at the "simplest" tori: 1D tori (finite gap solutions).
- 1D Cubic NLS $i\partial_t q = -\partial_{xx} q + |q|^2 q$, $x \in \mathbb{T}$, is integrable.
- It admits global Birkhoff coordinates:

$$\begin{aligned} \Phi : L^2(\mathbb{T}) \longrightarrow & \ \ell^2 \times \ell^2 \\ q \quad \longmapsto & \ (z_m, \bar{z}_m)_{m \in \mathbb{Z}}, \end{aligned}$$

  such that 1D cubic NLS becomes $i\dot{z}_m = \alpha_m(I) z_m$ where $I_m = |z_m|^2$ are the actions.
- It has twist.
- All solutions are periodic/quasi-periodic/almost-periodic.

1D invariant tori

- Consider one of the $\mathtt{d}$-dimensional quasiperiodic tori: Choose $\mathcal{S}_0 = (\mathtt{m}_1, \dots \mathtt{m}_\mathtt{d}) \subset \mathbb{Z}$ and a vector $I_\mathtt{m} = (I_{\mathtt{m}_1}, \dots, I_{\mathtt{m}_\mathtt{d}}) \in \mathbb{R}^\mathtt{d}_+$ and define

$$\mathtt{T}^\mathtt{d} = \left\{ (z_m)_{m \in \mathbb{Z}} : |z_{\mathtt{m}_i}|^2 = I_{\mathtt{m}_i}, i = 1, \dots, \mathtt{d},\ z_m = 0 \text{ if } m \notin \mathcal{S}_0 \right\}.$$

- Consider them as invariant objects for the 2D equation.
- For what time scales this torus is Lyapunov stable/unstable?
- Are there orbits in its neighborhood transfering energy (H^s norm explosion)?
- We will need to impose some (many) conditions on the torus (on $\mathcal{S}_0$ and $I_\mathtt{m}$).

Stability of the finite gaps solution

Theorem (A. Maspero – M. Procesi)

Fix $s \geq 1$. For a generic choice of support sites $\mathcal{S}_0$ there exists ε_0, T, K such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a Cantor set $\mathcal{I} \subset (0, \varepsilon)^{\mathtt{d}}$ such that the following holds true for any torus $\mathrm{T}^{\mathtt{d}} = \mathrm{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ with $I_{\mathtt{m}} \in \mathcal{I}$:

Fix any $\delta \ll 1$. Then, any solution of cubic defocusing NLS $u(t)$ such that

$$\mathrm{dist}_{H^s}\left(u(0), \Phi^{-1}(\mathrm{T}^{\mathtt{d}})\right) \leq \delta$$

satisfies

$$\mathrm{dist}_{H^s}\left(u(t), \Phi^{-1}(\mathrm{T}^{\mathtt{d}})\right) \leq K\delta \qquad \text{for all } |t| \leq T/\delta^2.$$

- These tori are Diophantine (and satisfy certain Melnikov conditions).

Some comments

- The same result is true for $s \in (0,1)$.

- Generic choice of $\mathcal{S}_0$: The modes in $\mathcal{S}_0$ cannot satisfy a finite number of relations (amount of relations only depending on $\mathtt{d}$).

- The set of "good" actions $\mathcal{I} \subset (0,\varepsilon)^{\mathtt{d}}$ has relative measure as close as $1/2$ as desired.

Main result

- Consider the same tori $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$.
- (Maybe ε_0 may change)

Theorem (M.G.– Z. Hani – E. Haus – A. Maspero – M. Procesi)

Fix $s \in (0,1)$ and $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ as before. Then, for any $\delta \ll 1$ and $K \gg 1$ there exists an orbit of cubic defocusing NLS and a time T such that

- $\mathrm{dist}_{H^s}\left(u(0), \Phi^{-1}(\mathtt{T}^{\mathtt{d}})\right) \leq \delta$
- $\|u(T)\|_{H^s} \geq K$.
- $|T| \lesssim e^{\left(\frac{K}{\delta}\right)^\beta}$ *for some* $\beta > 1$.

Comments and questions

- First instability result on quasiperiodic (Diophantine) tori in Hamiltonian PDEs.
- Transversal instability: these tori are stable as solutions of 1D NLS but there are 2D solutions arbitrarily close to the torus which attain huge Sobolev norms.
- Same result is valid for NLS on $\mathbb{T}^N$ with $N \geq 2$.
- $s \in (0, 1)$ implies that the obtained solution has very small mass but very large energy.
- Questions:
  - Why the torus needs to be small?
  - Why $\mathcal{S}_0$ generic?
  - Why not $s > 1$?
  - Why not full asymptotic measure?

Why the torus needs to be small? Why $\mathcal{S}_0$ generic?

- To understand the normal behavior at the torus: we reduce the linearized equation to constant coefficients.

- We only know how to do the reduction for small tori (linearized equation is close to constant coefficients $\to$ KAM scheme).

- To perform the reduction we need a generic choice of modes to avoid some resonances.

- Transversal instability "should be true" for non-small tori

Why $s \in (0,1)$?

- The tori we consider are Diophantine (satisfy several Melnikov conditions).
- Unstable orbits drift along resonances which are "far" from the tori.
- The bigger the s, the closer we have to be to the tori.
- For $s > 1$, we cannot use these resonances.
- In doing Birkhoff normal form around the torus we can kill more terms in H^s, $s > 1$, topology than for $s \in (0,1)$.

Why not full asymptotic measure?

- Even for a generic choice of modes, one encounters unavoidable resonances in reducing the linearized equation around the torus.
- Roughly speaking, for half measure in $(0,\varepsilon)^{\mathrm{d}}$ the matrix becomes hyperbolic and for the other half the matrix stays elliptic.
- To carry on our process, we need the matrix to be elliptic.
- Even if this hyperbolicity should be "good" for instabilities, not clear how to use it to achieve growth of Sobolev norms
(only involves Fourier modes in a vertical strip in $\mathbb{Z}^2$).

Sketch of the proof: Analysis of the linearization

- Express 1D NLS in Birkhoff coordinates
- Asymptotic estimates for the 1D eigenvalues of the linearization around the torus as $|m| \to \infty$.
- Procesi-Maspero: Reduce the linearization of 2D NLS around $\mathbb{T}^d$ to constant coefficients (KAM scheme, second order Melnikov conditions).
- Asymptotic estimates for the 2D eigenvalues of the linearization around the torus: For $n \in \mathbb{Z}$,

$$\Omega_n = i\left(n_1^2 + n_2^2 + \frac{\mathcal{O}(\varepsilon^2)}{\langle n_1\rangle^2} + \frac{\mathcal{O}(\varepsilon^2)}{\langle n_1\rangle^2 + \langle n_2\rangle^2}\right)$$

  for $n_2 \neq 0$ and $|n_1| \geq M_0$,
- Eigenvalues are very close to resonance for $|n_1| \gg 1$.

Sketch of the proof: Birkhoff normal form

- Third order Melnikov conditions hold for a positive measure set.
- Birkhoff normal form to eliminate completely the quadratic terms in the equation.
- Proof that certain fourth order Melnikov conditions hold for a positive measure set.
- Partial Birkhoff normal form to eliminate cubic non-resonant terms.
- Note that all these changes cannot be done in H^s (it will be large at the end)
- We do them in ℓ^1.
- Use the I-team approach: Use cubic resonant terms to achieve instabilities.

The I-team approach for solutions close to $u = 0$

$$-iu_t + \Delta u = |u|^2 u$$

- Cubic NLS as an ode (of infinite dimension) for the Fourier coefficients of u:

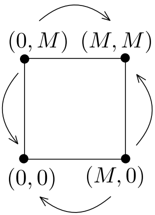
$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2 \\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Drift through resonances.
- Resonant monomial

$$n_1 - n_2 + n_3 - n = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$$

- Non-degenerate resonances form a rectangle in $\mathbb{Z}^2$.

Growth of Sobolev norms in a rectangle



- Take a solution on this rectangle with $M \gg 1$ such that
  - $a_{(M,0)}(0) = a_{(0,M)}(0) = 1$, $a_{(0,0)}(0) = a_{(M,M)}(0) = 0$
  - $a_{(M,0)}(T) = a_{(0,M)}(T) = 0$, $a_{(0,0)}(T) = a_{(M,M)}(T) = 1$

- Growth of Sobolev norms

$$\frac{\|u(T)\|_{H^s}^2}{\|u(0)\|_{H^s}^2} \sim \frac{|(0,0)|^{2s} + |(M,M)|^{2s}}{|(M,0)|^{2s} + |(0,M)|^{2s}} = \frac{(\sqrt{2}M)^{2s}}{M^{2s} + M^{2s}} \sim 2^{s-1}$$

- We "pump" mass from $(M,0)$ and $(0,M)$ to $(0,0)$ and (M,M).
- We "pump" the Sobolev norm to (M,M).

Traveling through rectangles

- One needs to concatenate many rectangles to attain a bigger growth.
- Number of concatenations: $N \sim \log(K/\delta) \gg 1$.
- At each step, the Sobolev norm is pushed to half of the modes (the ones further out)
- We need many modes to be able to push Sobolev norm through the N "generations".
- They consider a finite set of modes $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N \subset \mathbb{Z}^2$ of large size $|\Lambda_j| = 2^{N-1}$, $N \sim \log(K/\delta)$.

Traveling through rectangles

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N \subset \mathbb{Z}^2, \qquad |\Lambda_j| = 2^{N-1}, \qquad N \sim \log(K/\delta)$$

- They choose carefully Λ such that the modes interact in a very particular way.
- Each rectangle has modes only in two consecutive generations.
- Each mode in generation Λ_j pumps energy from a rectangle involving modes in Λ_{j-1} to a rectangle involving modes in Λ_{j+1}.

The I-team approach for the cubic case

- After further reductions: finite dimensional (toy) model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right), \; j = 1, \dots N.$$

  which approximates well certain solutions of NLS.

- Each b_j represents the 2^{N-1} modes in Λ_j.

- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.

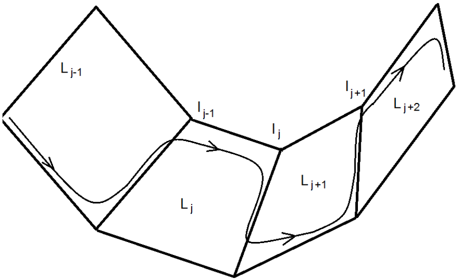
Dynamics of the cubic toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \ldots N,$$

- Each 4-dimensional plane

$$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

  is invariant and corresponds to two generations interacting
- Namely, interactions through rectangles.
- In L_j, $\mathbb{T}_j = \{b_j \neq 0, b_{j+1} = 0\}$ and $\mathbb{T}_{j+1} = \{b_j = 0, b_{j+1} \neq 0\}$ are (partially hyperbolic) periodic orbits.
- They are connected through a heteroclinic orbit



- I-team: To travel close to L_j from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$, they shadow this heteroclinic.
- M. G. – V. Kaloshin: Shadowing with time estimates.
- Delshams-Simon study the shadowing with Shilnikov techniques.
- Conclusion: Orbits which are localized first at b_1 and after some time at b_N.

The I-team approach for solutions close a torus

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- After normal form we have an eq. "similar" to the original NLS.
- Problems:
  - Eigenvalues Ω_n are not to $|n|^2$, but close to it for modes in Λ.
  - We drift along almost-resonances.
  - This creates extra error terms which blow up for large times.
  - Coefficients of the cubic monomials are different in each monomial: They come from the interaction between the torus and the modes in the monomial.

The I-team approach for solutions close a torus

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2 \\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Goal: Choose $\Lambda \subset \mathbb{Z}^2$ to "minimize" this interaction.
- For a well chosen $\Lambda \subset N\mathbb{Z}\times N\mathbb{Z}$, the coefficients of the associated monomials are
  $$1+\mathcal{O}(N^{-1})$$
  thanks to: Smoothing properties of the reducibility change of coordinates + Combinatorial analysis of the Birkhoff NF changes of coordinates.
- Treating the extra terms as errors, one can carry the I-team approach.
- Toy model leads to growth of Sobolev norms after incorporating the errors and undoing all the changes of coordinates.