Incompressible Elasticity in 2D

Zhen(震)  Lei(雷)

Fudan University
Joint with Thomas C. Sideris and Yi Zhou
Contents

✓ Incompressible Elasticity

✓ The Key Question and Its Difficulties
  ★ some previous progress

✓ Almost Global Well-posedness of Small Solutions in 2D
Notations

The flow map $X(t, y)$:

It maps the material point $y \in \Omega_0$ at time $t = 0$ to the space position $x = X(t, y) \in \Omega_t$ at time $t$. $(t, y)$ will be called Lagrangian coordinate, while $(t, x)$ Euler
The flow map $X(t, y)$ generates a velocity field $v$, which, at time $t$ and spatial position $x$, is given by:

$$v(t, x) = \frac{\partial X(t, y)}{\partial t} \bigg|_{y=X^{-1}(t,x)}.$$

Alternatively, one may also think that a given velocity field $v(t, x)$ generates the flow map by solving:

$$\frac{dX(t, y)}{dt} = v(t, x) \bigg|_{x=X(t,y)}, \quad X(0, y) = y.$$
For perfect fluid flows, the dynamics is determined by the following Lagrangian functional, which is related to the associated the kinetic energy:

\[ \mathcal{L}(X; T, \Omega) = \frac{1}{2} \int_{0}^{T} \int_{\Omega} |v(t, X(t, y))|^2 dy dt. \]

It is known that the first variation of \( \mathcal{L}(X) \), under the incompressibility constraint, gives the well-known Euler equation.
Motion of elastic materials is also determined by their elastic energies. Define the deformation gradient $F(t, x)$ by:

$$F(t, X(t, y)) = \frac{\partial X(t, y)}{\partial y}. \quad (1)$$

Incompressibility means volume-preserving. In mathematics, that is

$$\det F \equiv 1 \quad (2)$$

since $\int_U dy \equiv \int_{X(t,U)} dX$ for any domain $U$. 

Incompressible Elasticity in 2D – p. 6/44
Consider the most basic storage energy functionals

$$\hat{W}(X(t, x)) = W(F(t, x))$$

For isotropic materials, $W$ depends on $F$ only in terms of the invariants of $F^\top F$. In 2D, those are trace and determinant.

Perfect fluids: $W = W(\det F^\top F)$.

Hookean elastic case $W = \frac{1}{2}|F|^2$. 
The Lagrangian function in this case is

\[ \mathcal{L}(X; T, \Omega) = \int_0^T \int_\Omega \frac{1}{2} |X_t(t, y)|^2 \]

\[ - \frac{1}{2} |F(t, X(t, y))|^2 + p(t, y)(\det F - 1) \, dy \, dt. \]

Here \( p(t, y) \) is a Lagrangian multiplier which is responsible for the incompressibility, which is equivalent to

\[ \nabla \cdot \nu = 0. \]
E-L equation:

\[ X_{tt} - \Delta_{y}X + F^{-T}\nabla_{y}p = 0. \]

The incompressibility constraint:

\[ \det \nabla X = 1. \]
Key Question

Key Question: To solve the flow map $X(t, \cdot)$, or equivalently, to solve the above incompressible elastic system.

We will formulate it in Euler coordinate: quasi-linear wave type equation. Current interests center around small-data global regularity.
Vector Fields

Suppose that $X, p$ is a critical point of $\mathcal{L}$. If we define

$$\tilde{X}(t, y) = Q(s)X(t, Q^\top(s)y), \quad \tilde{p}(t, y) = p(...),$$

where

$$Q(s) = e^{sA}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Then $\tilde{X}, \tilde{p}$ is also critical point of $\mathcal{L}$. This invariance group gives that

$$\left(\frac{\partial \Omega X}{\partial y}\right)^\top (\partial_t^2 - \Delta_y)X + \left(\frac{\partial X}{\partial y}\right)^\top (\partial_t^2 - \Delta_y)\Omega X + \nabla_y \Omega p = 0.$$
Vector Fields

Similarly, one can derive that

\[
\left(\frac{\partial \tilde{S}X}{\partial y}\right)^\top \left(\partial_t^2 - \Delta_y\right)X + \left(\frac{\partial X}{\partial y}\right)^\top \left(\partial_t^2 - \Delta_y\right)\tilde{S}X + \nabla_y S p = 0.
\]

where

\[
S = t \partial_t + r \partial_r, \quad \tilde{S} = S - 1.
\]

and

\[
\Omega X = \partial_\theta X + AX, \quad \Omega p = \partial_\theta p.
\]

Unfortunately, there is no Lorentz invariance.
Incom-Elasiticity in Euler Chart

Incompressible Elasticity in Euler coordinate:

\[
\begin{align*}
\begin{cases}
vt + v \cdot \nabla v + \nabla p &= \nabla \cdot (FF^T), \\
F_t + v \cdot \nabla F &= \nabla v F, \\
\nabla \cdot v &= 0.
\end{cases}
\end{align*}
\]

Make use of the dispersive nature by studying small

\[
(G, v) = (F - I, v).
\]
Connection to Other System

✓ Add $\Delta v \implies$ Viscoelasticity
✓ Ignore elastic force $\implies$ Euler or Navier-Stokes
✓ By $\nabla \cdot F^\top = 0$, one may assume that $F = (\nabla^\perp \phi)^\top$. Then

$$
\begin{cases}
  v_t + v \cdot \nabla v + \nabla \tilde{p} = -\nabla \cdot (\nabla \phi \otimes \nabla \phi), \\
  \phi_t + v \cdot \nabla \phi = 0, \\
  \nabla \cdot v = 0.
\end{cases}
$$

MHD: $\phi$ is a scalar.
Main Difficulties

Linearization:

\[ \nu_{tt} - \Delta \nu = 0, \quad G_{tt} - \Delta G = \nabla \times (\nabla \times G). \]

If \( \nabla \times (\nabla \times G) \) can be treated as a forcing term, then the main part of the linearized system is of wave type. Fortunately, this is true because (thesis of L.)

\[ \nabla \times G = Q(G, \nabla G). \]
Main Difficulties

So the key points for global or long time existence are

✓ dimension, which determines the time decay rate

✓ null structure of nonlinearies, which gives nonresonance along the light cone
Main Difficulties

In general, energy estimate gives (quadratic non)

\[
\frac{dE_s(t)}{dt} \lesssim \|D^{s-2}v\|_{L^\infty} E_s(t).
\]

Decay type estimate gives

\[
\|D^{s-2}v(t, \cdot)\|_{L^\infty} \lesssim \frac{\sqrt{E_s}}{(1 + t)^\alpha}.
\]

- $\alpha > 1$: subcritical
- $\alpha = 1$: critical
- $\alpha < 1$: supercritical
Main Difficulties in 2D

Let $S = t \partial_t + r \partial_r$ be the scaling operator,
$\Omega_{ij} = x_i \partial_j - x_j \partial_i$ rotation and
$L_j = \Omega_{0j} = t \partial_j + x_j \partial_t$ Lorentz.

Theorem 1 (Klainerman). Weighted inequality:

$$|u(t, x)| \lesssim \sum_{|\alpha| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1} \left\| \Gamma^\alpha u(t, \cdot) \right\|_{L_x^2} \frac{1}{(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{1}{2}}}.$$
Main Difficulties in 2D

Hence,

- $n \geq 4$ subcritical: Global well-posedness (WP)
- $n = 3$ critical: Global WP under null condition, by Klainerman (86), Christodoulou (86).
- $n = 2$ supercritical: Global WP under double null conditions, Alinhac (01)
Main Difficulties in 2D

The elastic system is much more involved.

✓ Two different propagation speeds
 ✓ Null structure is hard to use.
Main Difficulties in 2D

Progress in 3D

✓ 3D compressible case: Sideris (97, 00), Agemi (00)

✓ 3D incompressible case: Sideris and Thomases (05, 06, 07)
Main Difficulties in 2D

Open questions:

✓ Wave systems with different speeds, 2D

✓ Compressible Elasticity, 2D

✓ Incompressible Elasticity, 2D
Main Result

Our main theorem is:

**Theorem 2** (L., Sideris and Zhou, 12). *For sufficiently small initial data, the 2D incompressible elasticity is almost global well-posed.*

Here *almost global* means that if the norm of the initial data is of $\varepsilon$-order, then the lifespan of the solution is at least $\exp(C_0/\varepsilon)$ for some $C_0 > 0$. 
Viscoelasticity

With a viscous term $\Delta u$ in momentum equation:

- √ 2D: Lin-Liu-Zhang (05), L. Zhou (05)
- √ 3D: L.-Liu-Zhou (08)
- √ 2D small strain: L.-Liu-Zhou (08), L. (10, 13)
- √ Survey: Lin (12)
Main Difficulties in 2D

In 2D elasticity, the difficulties

- dimension 2, the time decay rate $\frac{1}{\sqrt{1+t}}$ is supercritical
- structure of nonlinearities
- nonlocal nature
Vector Fields in Euler

Motivated by the invariance property of this equation in Lagrangian coordinate, we have

\[
\begin{align*}
\partial_t \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G &= -\nabla \Gamma^\alpha p + \sum_{\beta+\gamma=\alpha} \Gamma^\beta v \cdot \nabla \Gamma^\gamma v + \nabla \cdot (\Gamma^\gamma G \Gamma^\beta G^T) \triangleq f_\alpha, \\
\partial_t G - \nabla \cdot \Gamma^\alpha G &= \sum_{\beta+\gamma=\alpha} \nabla \Gamma^\beta v \Gamma^\gamma G - \Gamma^\gamma v \cdot \nabla \Gamma^\beta G \triangleq g_\alpha, \\
\nabla \cdot \Gamma^\alpha v &= 0.
\end{align*}
\]

Here $\Gamma$ be any of $\{\partial_t, \partial_1, \partial_2, \Omega, S\}$. 

Incompressible Elasticity in 2D – p. 26/44
Proof

The modified rotation operator:

\[ \Omega f = \begin{cases} 
\partial_{\theta} f, & f \text{ scalar}, \\
\partial_{\theta} f + Af, & f \text{ vector}, \\
\partial_{\theta} f + [A, f], & f \text{ matrix}.
\end{cases} \]

We often use the decomposition:

\[ \nabla = \omega \partial_r + r^{-1} \omega_{\perp} \partial_{\theta}. \]
Proof

Based on the structures, we have:

\[ \nabla \cdot \Gamma^\alpha G^\top = 0 \]

and

\[ \nabla^\perp \cdot \Gamma^\alpha G = h_\alpha, \]

where

\[
(h_\alpha)_i = \sum_{\beta+\gamma=\alpha} \left[ \Gamma^\beta G_{m1} \partial_m \Gamma^\gamma G_{i2} - \Gamma^\beta G_{m2} \partial_m \Gamma^\gamma G_{i1} \right].
\]
Proof

Define the generalized energy by

$$E_k(t) = \sum_{|\alpha| \leq k} \| \Gamma^\alpha(v, G) \|^2_{L^2}. \quad (2)$$

We also define the weighted energy norm

$$X_k(t) = \sum_{|\alpha| \leq k-1} \| < t - r > \nabla \Gamma^\alpha(v, G) \|_{L^2}. \quad (3)$$
Proof

Structures:

✓ Due to the incompressibility
\[ \nabla \cdot \mathbf{v} = 0 = \nabla \cdot F^T, \]
the following are good unknowns near the light cone \( r = t \):

\[ \mathbf{v} \cdot \omega, \quad G^T \omega \quad (\omega = x/r). \]
Proof

Structures:

✓ By identity (L.-Liu-Zhou, 08)

\[ \partial_j G_{ik} - \partial_k G_{ij} = G_{mk} \partial_m G_{ij} - G_{mj} \partial_m G_{ik}, \]

the following is a good unknown near the light cone \( r = t \):

\[ G_{\omega^\perp}. \]
Proof

Structures: An extra intrinsic good unknown is

\[ \nu + G\omega \]

This can be seen via Alinhac’s ghost weight method, which was the first time to be applied for nonlocal problem.
Proof

The pressure satisfies null condition.

**Lemma 3 (Estimate of pressure).** We have

\[ \| \nabla \Gamma^\alpha p \|_{L^2} \lesssim \| f_\alpha \|_{L^2} \]

\[ \| \nabla \Gamma^\alpha p \|_{L^2} \lesssim \sum_{\beta + \gamma = \alpha \atop |\beta| \leq |\gamma|} \| \partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk} \|_{L^2}, \]

for all \( |\alpha| \leq k - 1. \)
Proof

Lemma 4 (Structures). Define

\[
\begin{align*}
L_k &= \sum_{|\alpha| \leq k} \left[ |\Gamma^\alpha v| + |\Gamma^\alpha G| \right], \\
N_k &= \sum_{|\alpha| \leq k-1} \left[ t \left( |f_\alpha| + |g_\alpha| + |\nabla \Gamma^\alpha p| \right) + (t + r)|h_\alpha| \right].
\end{align*}
\]

Then for all $|\alpha| \leq k - 1$ (First two B-T-L),

\[
\begin{align*}
& r |\partial_r \Gamma^\alpha v \cdot \omega| \sim L_k \\
& r |\partial_r \Gamma^\alpha G^\top \omega| \sim L_k \\
& r |\partial_r \Gamma^\alpha G \omega^\perp| \sim L_k + N_k.
\end{align*}
\]
Proof

**Lemma 5 (Better Decay of Good Unknowns near Light Cone).** For $|\alpha| \leq k - 2$, we have

$$\|r \Gamma^\alpha v \cdot \omega\|_{L^\infty} + \|r \Gamma^\alpha G^\top \omega\|_{L^\infty} \lesssim E^{1/2}_{|\alpha|+2}.$$

*Proof.* Sobolev imbedding on sphere + incompressibility.

□
Proof

Lemma 6 (Structures). Recall that

\[
\begin{align*}
L_k &= \sum_{|\alpha| \leq k} \left[ |\Gamma^\alpha v| + |\Gamma^\alpha G| \right], \\
N_k &= \sum_{|\alpha| \leq k-1} \left[ t (|f_\alpha| + |g_\alpha| + |\nabla \Gamma^\alpha p|) + (t + r)|h_\alpha| \right].
\end{align*}
\]

For all $|\alpha| \leq k - 1$,

\[ (t \pm r) |\nabla \Gamma^\alpha v \pm \nabla \cdot \Gamma^\alpha G \otimes \omega| \lesssim L_k + N_k. \]
Proof

Take a look at the proof which seems not transparent: Using \( S = t \partial_t + r \partial_r \) and the equation:

\[
t \nabla \Gamma^\alpha v + r \partial_r \Gamma^\alpha G = S \Gamma^\alpha G - t g_\alpha
\]

\[
t \nabla \cdot \Gamma^\alpha G + r \partial_r \Gamma^\alpha v = S \Gamma^\alpha v - t f_\alpha + t \nabla \Gamma^\alpha p.
\]

This is rearranged as follows:

\[
t \nabla \Gamma^\alpha v + r \nabla \cdot \Gamma^\alpha G \otimes \omega = r [\nabla \cdot \Gamma^\alpha G \otimes \omega - \partial_r \Gamma^\alpha G]
\]

\[+ S \Gamma^\alpha G - t g_\alpha\]

\[
t \nabla \cdot \Gamma^\alpha G \otimes \omega + r \nabla \Gamma^\alpha v = r [\nabla \Gamma^\alpha v - \partial_r \Gamma^\alpha v \otimes \omega]
\]

\[+ [S \Gamma^\alpha v - t f_\alpha + t \nabla \Gamma^\alpha p] \otimes \omega.
\]
Proof

**Lemma 7** (Estimate of Nonlinearities Using Weighted Energy). We have

\[ \| N_k(t) \|_{L^2} \lesssim E_k(t) + E_k(t)^{1/2} X_k(t)^{1/2}. \]

*Proof.* Away from the light cone, using weighted energy. Near the light cone, using the better estimate for good unknowns. \(\square\)
Proof

Lemma 8 (Estimate of Weighted Energy). If \( E_k(t) \ll 1 \), then \( X_k(t) \lesssim E_k(t)^{1/2} \).

Proof: By structures, the main contribution of \( X_k(t)^2 \) is \( E_k \) and

\[
\sum_{|\alpha| \leq k-1} \left[ \| (t - r) \nabla \Gamma^\alpha v \|^2_{L^2} + \| (t - r) \nabla \cdot \Gamma^\alpha G \|^2_{L^2} \right].
\]
Proof

Then use *structures* and the decomposition:

\[
\nabla \Gamma^\alpha \nu = \frac{1}{2} [ \nabla \Gamma^\alpha \nu + \nabla \cdot \Gamma^\alpha G \otimes \omega ] \\
+ \frac{1}{2} [ \nabla \Gamma^\alpha \nu - \nabla \cdot \Gamma^\alpha G \otimes \omega ]
\]  

and

\[
\nabla \cdot \Gamma^\alpha G = \frac{1}{2} [ \nabla \Gamma^\alpha \nu + \nabla \cdot \Gamma^\alpha G \otimes \omega ] \omega \\
- \frac{1}{2} [ \nabla \Gamma^\alpha \nu - \nabla \cdot \Gamma^\alpha G \otimes \omega ] \omega,
\]
Lemma 9 (Further Better Decay of Good Unknowns near Light Cone). Let $k \geq 4$, $E_k \ll 1$, $\omega = x/|x|$. Then we have

$$
\| r(\partial_r \Gamma^\alpha v + \partial_r \Gamma^\alpha G\omega) \|_{L^2} + \| r \partial_r \Gamma^\alpha G\omega^\perp \|_{L^2} \lesssim E_{|\alpha|+1}^{1/2},
$$

$$
\| r(\Gamma^\alpha v + \Gamma^\alpha G\omega) \|_{L^\infty} + \| r \Gamma^\alpha G\omega^\perp \|_{L^\infty} \lesssim E_{|\alpha|+1}^{1/2}.
$$

Proof. Using the equations and structures. □

Remark 10. This is as good as linear wave equations.
Proof

Now we are ready to derive a critical energy estimate near the light cone, based on a very delicate estimate on pressure and nonlinearities, and the application of a ghost weight method by Alinhac. The critical energy estimate away from the light cone is due to the Klainerman-Sideris’s weighted $L^2$ energy estimate.
Proof

The final estimate:

\[ \tilde{E}_k'(t) \leq C_0(1 + t)^{-1}\tilde{E}_k(t)^{3/2}, \quad 0 \leq t < T. \]

Here \( E_k \sim \tilde{E}_k \). This implies that \( E_k(t) \) remains bounded by \( 2\epsilon^2 \) on a time interval of order \( T \sim \exp(C_0/\epsilon) \).
Thank you very much!!

谢谢关注!!