All order asymptotics for $\beta$-ensembles in the multi-cut regime

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joint work
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All order asymptotics for $\beta$-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Ideas about the proof

4. Perspectives
The 1d beta-ensemble is ...

- The probability measure on $A^N \subseteq \mathbb{R}^N$

$$d\mu_N^A = \frac{1}{Z_N^A} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-N(\beta/2)V(\lambda_i)} 1_A(\lambda_i) d\lambda_i \quad \beta > 0$$

- It is the measure induced on eigenvalues of a random matrix $M$

$$dM e^{-N(\beta/2) \text{Tr} V(M)}$$

Wigner, Dyson, Mehta (50s-60s)

$$\begin{cases} 
\beta = 1 & \text{real symmetric matrices} \\
\beta = 2 & \text{hermitian matrices} \\
\beta = 4 & \text{quaternionic self-dual matrices}
\end{cases}$$

$M = \text{triagonal}$

all $\beta > 0$, $V$ quadratic

Dumitriu, Edelman ’02
We would like to study when $N \to \infty$ ...

- the (random) empirical measure

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

Example: what kind of random variable is $\sum_{i=1}^{N} f(\lambda_i)$?

- the partition function

$$Z^A_N = \int_{A^N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^{N} e^{-N(\beta/2)V(\lambda_i)} \, d\lambda_i$$
The leading order ... is given by a continuous approximation

Classical result

Assume $V$ continuous and confining \((\liminf_{|x|\to\infty} \frac{V(x)}{2 \ln |x|} > 1)\)

- \(\mathcal{E}[\mu] = \iint \ln |x - y| d\mu(x) d\mu(y) - \int V(x) d\mu(x)\)
  has a unique maximizer \(\mu_{eq} \in \mathcal{M}^1(A)\) characterized by
  \(\exists C \quad 2 \int_A \ln |x - y| d\mu_{eq}(y) - V(x) \leq C\) with equality \(\mu_{eq}\) everywhere

- \(L_N \to \mu_{eq}\) almost surely and in expectation

- \(Z^A_N = \exp \left\{ N^2 (\beta/2) (\mathcal{E}[\mu_{eq}] + o(1)) \right\}\)
Large deviations (local result)

- $\lambda_i$'s feel the effective potential

$$V_{\text{eff}}(x) = V(x) - 2 \int \ln |x - y| \, d\mu_{\text{eq}}(y) - C \geq 0$$

- For any closed $F \subseteq A$

$$\text{Prob}^A_N[\exists i \; \lambda_i \in F] \leq \exp \left( -N(\beta/2) \left\{ \min_{x \in F} V_{\text{eff}}(x) + o(1) \right\} \right)$$

$\sim$ One can restrict to a compact $B \subseteq A$ neighborhood of $\{V_{\text{eff}}(x) = 0\}$

$$Z_N^B = Z_N^A (1 + o(e^{-cN}))$$
Large deviations (global result)

- \( \mathcal{D}[\mu_1, \mu_2] = - \int \ln |x - y| d(\mu_1 - \mu_2)(x) d(\mu_1 - \mu_2)(y) = \int_0^\infty \frac{|\text{FT}[\mu_1 - \mu_2](k)|^2}{k} \)

defines a distance \( \in [0, +\infty] \) on \( \mathcal{M}^1(A) \)

such that \( \left| \int f(x) d(\mu_1 - \mu_2)(x) \right| \leq \sqrt{2} \left( \int_{\mathbb{R}} k |\text{FT} f(k)|^2 dk \right)^{1/2} \mathcal{D}^{1/2}[\mu_1, \mu_2] \)

- Let us pick a nice regularization \( L_N = \frac{1}{N} \sum_{i=1}^N \delta_{\chi_i} \rightsquigarrow \tilde{L}_N \)

**Proposition**

Borot, Guionnet (’11)

If \( V \) is \( C^3 \), we have for \( N \) large enough

\[
\text{Prob}^A_N [\mathcal{D}^{1/2}[\tilde{L}_N, \mu_{eq}] \geq t] \leq \exp \left( CN \ln N - N^2 (\beta/2)t^2 \right)
\]
More on the equilibrium measure ... 

- \( V \) real-analytic \( \implies \left\{ \begin{array}{l} \mu_{\text{eq}} \text{ is supported on a finite number of segments} \\ S = \bigcup_{h=0}^{g} [a_h, b_h] \end{array} \right. 

- \( \alpha \in \partial S \) is a hard edge if \( \alpha \in \partial A \), is a soft edge otherwise

\[
d\mu_{\text{eq}}(x) = \frac{1_S(x)dx}{2\pi} M(x) \prod_{\alpha \text{ soft}} |x - \alpha|^{1/2} \prod_{\alpha \text{ hard}} |x - \alpha|^{-1/2}
\]

We say that \( \mu_{\text{eq}} \) is off-critical when \( M(x) > 0 \) on \( A \).
Finite size corrections: we assume ...

- $V = V_0 + (1/N)V_1 + \cdots$
  - $V_0$ real analytic on $A$
  - $V_1$ complex analytic on $A$

- Control of large deviations
  - $V_{\text{eff}}(x) > 0$ for $x \in A \setminus S$

- $\mu_{\text{eq}}$ is off-critical

- $f$ = test function, analytic on $A$
Result in the 1-cut regime

1/N asymptotic expansion

\[ Z_N^A = N^{\gamma N + \gamma'} \exp \left[ \sum_{k \geq -2} N^{-k} F_k + O(N^{-\infty}) \right] \]

\[ \gamma, \gamma' \text{ depend only on } \beta \text{ and the nature of the edges} \]

\[ F_k = \sum_{h=0}^{[k/2]+1} \left( \frac{\beta}{2} \right)^{1-h} \left( 1 - \frac{2}{\beta} \right)^{k+2-2h} F_{[h];k+2-2h} \]

Central limit theorem

\[ \left( \sum_{i=1}^{N} f(\lambda_i) - N \int_A f(\xi) d\mu_{eq}(\xi) \right) \rightarrow \text{ (non-centered) gaussian} \]
Result in the \((g + 1)\)-cuts regime

- Oscillatory asymptotic expansion

\[
Z_N^A = N^{\gamma N + \gamma'} (D_N \Theta_{-N \epsilon_\ast}) (F'_1 | F''_2) \exp \left[ \sum_{k \geq -2} N^{-k} F_k + O(N^{-\infty}) \right]
\]

where \( D_N = \sum_{r \geq 0} \frac{1}{r!} \sum_{\ell_1, \ldots, \ell_r \geq 1 \atop k_1, \ldots, k_r \geq -2 \atop \sum_i (k_i + \ell_i) > 0} N^{-\left( \sum_i k_i + \ell_i \right)} \prod_{i=1}^{r} \frac{F_{k_i}^{(\ell_i)} \cdot \nabla \otimes \ell_i}{\ell_i!} \)

acts as a differential operator on the Siegel theta function

\[
\Theta_\mu(w|Q) = \sum_{m \in \mathbb{Z}^g} e^{w \cdot (m+\mu) + \frac{1}{2} (m+\mu) \cdot Q \cdot (m+\mu)}
\]

- Moving characteristics \( \mu = -N \epsilon_\ast \mod \mathbb{Z}^g \)

Quadratic form \( Q = F''_2 = 2i\pi (\beta/2) \times \text{(period matrix)} < 0 \)
No central limit theorem in general ...

\[ \mathbb{E} \left[ e^{i s \left( \sum_{i=1}^{N} f(\lambda_i) - N \int f(x) d\mu_{eq}(x) \right)} \right] \sim e^{i s m_1[f] - m_2[f]s^2/2} \frac{\Theta - N \epsilon^* \left( F'_1 + i s v[f]|F''_2 \right)}{\Theta - N \epsilon^* \left( F'_1|F''_2 \right)} \]

(non-centered) gaussian

+ discrete Gaussian, centered at \( \mu = -N \epsilon^* \mod \mathbb{Z}^g \)

\[ v[f] \propto \left( \int_S \frac{f(x) x^i dx}{\prod_{\alpha} |x - \alpha|^{1/2}} \right)_{0 \leq i \leq g-1} \]

Corollary

\[ \left( \sum_{i=1}^{N} f(\lambda_i) - N \int_A f(\xi) d\mu_{eq}(\xi) \right) \]

converges in law along subsequences
If $1/N$ expansion exists, then

$$Z_N = N^{\gamma N + \gamma'} \exp \left( \sum_{h \geq 0} N^{2-2h} F[h] \right)$$

and $F[h]$ can be computed by the moment method

Ambjørn, Chekhov, Kristjansen, Makeenko, 90s

Rewriting of $F[h]$ in terms of a universal topological recursion

Eynard, ’04

Existence of $1/N$ expansion by
- analysis of SD equations Albeverio, Pastur, Shcherbina ’01
- RH techniques Ercolani, McLaughlin ’02
- analysis of int. system Bleher, Its, ’05
if $1/N$ expansion exists, then

$$F_k = \sum_{h=0}^{\lceil k/2 \rceil + 1} \left( \frac{\beta}{2} \right)^{1-h} \left( 1 - \frac{2}{\beta} \right)^{k+2-2h} F_{[h];k+2-2h}$$

and $F_{[h];m}$ computed by a $\beta$-topological recursion

Chekhov, Eynard '06

Central limit theorem

Johansson '98

Existence of $1/N$ expansion (analysis of SD eqn)

Borot, Guionnet '11
History in the \((g + 1)\)-cuts regime

\(\beta = 2\)

- numerous observations of oscillatory behavior
  - physicists, ‘90s

- asymptotics of \(\langle \det(x - M) \rangle_{N \times N}\) up to \(o(1)\) (RH techniques)
  - Deift, Kriecherbauer, McLaughlin, Venakides, Zhou ’99

- heuristic derivation up to \(o(1)\)
  - Bonnet, David, Eynard ’00

- generalization to all orders
  - Eynard ’07

- observation of “no CLT”
  - Pastur ’06

\(\beta > 0\)

- Proof of “no CLT” and asymptotics of \(Z^A_N\) up to \(o(1)\)
  - Shcherbina ’12

- General proof
  - Borot, Guionnet ’13
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Orthogonal polynomials and random matrices

For $\beta = 2$

measure over the space of $N \times N$ hermitian matrices

is the $N$th orthogonal polynomial for the weight $dx \, e^{-n V(x)}$ on $\mathbb{R}$

- Let $h_{N,n} = \text{norm of } P_{N,n}$

$satisfies a 3-term recurrence relation$

$$ (x - \beta_{N,n}) \hat{P}_{N,n}(x) = \sqrt{h_{N,n}} \hat{P}_{N+1,n}(x) + \sqrt{h_{N-1,n}} \hat{P}_{N-1,n}(x) $$
The coefficients are solutions of a Toda chain:

\[
\begin{align*}
\frac{\partial}{\partial t_1} u_{N,n} &= v_{N,n} - v_{N-1,n} \\
\frac{\partial}{\partial t_1} v_{N,n} &= e^{u_{N+1,n}} - e^{u_{N,n}}
\end{align*}
\]

\[V(x) = V_0(x) + \sum_{j \geq 1} \frac{t_j}{j} x^j\]

\[
\frac{1}{Z_{N,n}} \, dM \, e^{-n \text{Tr} \, V(M)}
\]

- The coefficients are solutions of a Toda chain:
- \(\partial_{t_j}\) are the higher Toda flows
- Initial condition prescribed by the string equations
- \(Z_{N,n} = N! \prod_{j=0}^{N-1} h_{j,n}\) is the Tau function
The continuum limit of Toda

\[ N, n \to \infty \quad N/n = t \text{ fixed} \]

- if the model with \( V/t \) has \((g+1)\)-cuts and is off-critical

main result & \[ h_{N,n} = \frac{1}{N + 1} \frac{Z_{N+1,nN/(N+1)}}{Z_{N,n}} \]

\[ u_{N,n} = \ln h_{N,n} \]

\[ V(x) = \frac{x^2}{2} + hG \frac{x^4}{4} + h^2 \frac{x^6}{6} \]

\[ h = 0.05 \]

2 cuts

3 cuts

\[ N = 100 \text{ solid line} \]

from Jurkiewicz '91 Phys. Lett. B, 261, 3
Asymptotics of orthogonal polynomials

\[ N, n \to \infty \]
\[ N/n = t \quad \text{fixed} \]

- **main result** + \[ P_{N,n}(x) = \frac{Z^V N,n (1/N) \ln(x-\bullet)}{Z^V N,n} \]

\[ \implies \text{all-order asymptotics of } P_{N,n}(x) \text{ for } x \text{ away from its zero locus} \]

- \( \beta = 1, 4 \) are related to skew orthogonal polynomials/Pfaff lattice

\[ \langle P_{j,n} | P_{k,n} \rangle = (\delta_{j,k-1} - \delta_{j-1,k})h_{j,n} \]

\[ \begin{cases} 
M = \text{real symmetric} \\
\beta = 1 \\
\langle f | g \rangle_{\beta=1} = \int_{\mathbb{R}^2} dx dy e^{-n(V(x)+V(y))} \sgn(x-y) f(x)g(y) \\
N_{\beta=1} = 2N 
\end{cases} \]

\[ \begin{cases} 
M = \text{quaternionic self-dual} \\
\beta = 4 \\
\langle f | g \rangle_{\beta=4} = \int_{\mathbb{R}} dx e^{-nV(x)} (f(x)g'(x) - f'(x)g(x)) \\
N_{\beta=4} = N 
\end{cases} \]

\[ P_{2N,n}(x) = \mathbb{E}_{N_{\beta} \times N_{\beta}} \left[ \det(x - M) \right] \]
\[ P_{2N+1,n}(x) = \mathbb{E}_{N_{\beta} \times N_{\beta}} \left[ (x + \text{Tr } M) \det(x - M) \right] \]

\[ \implies \text{similar asymptotic results} \]
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Conditioning on the filling fractions

- From local large deviations: up to $o(e^{-cN})$, we can choose

$$A = \bigcup_{h=0}^{g} A_h$$

- We will study $\mu_{(N_0, \ldots, N_g)}^{(A_0, \ldots, A_g)} = \mu_N^A$ conditioned to have

$$\begin{cases} N_0 \text{ first } \lambda \text{'s in } A_0 \\ N_1 \text{ next } \lambda \text{'s in } A_1 \\ \text{etc.} \end{cases}$$

The partition function decomposes

$$Z_N^A = \sum_{N_0 + \cdots + N_g = N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0, \ldots, N_g)}^{(A_0, \ldots, A_g)}$$
Correlators and partition function

- We will show a $1/N$ expansion for the $m$-point correlators:

$$ W_m(x_1, \ldots, x_m) = \mu_N^A - \text{cumulant} \left( \sum_{i_1=1}^{N} \frac{1}{x_1 - \lambda_{i_1}}, \ldots, \sum_{i_m=1}^{N} \frac{1}{x_m - \lambda_{i_m}} \right) $$

$x_i \in \mathbb{C} \setminus A$

- If $(V_t)_t$ is a smooth family of potentials respecting our assumptions

$$ \frac{Z_N^A; V_1}{Z_N^A; V_0} = \exp \left[ -N(\beta/2) \int_A \frac{dx}{2\pi i} \partial_t V_t(x) W_1^{V_t}(x) \right] \text{ will have a large } N \text{ expansion}$$

- We need a reference $V_0$ where $Z_N^A; V_0$ can be exactly computed
The Schwinger-Dyson equations

- Integration by parts $\implies$ exact relations between $\mu^A_N$-cumulants

$$\int \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} e^{-N(\beta/2)V(\lambda_i)} d\lambda_i$$

- E.g:

$$\mu^A_N \left[ \sum_{i=1}^{N} \frac{1}{(x - \lambda_i)^2} + \sum_{1\leq i<j\leq N} \frac{\beta}{(x - \lambda_i)(x - \lambda_j)} - \frac{N\beta}{2} \sum_{i=1}^{N} \frac{V'(\lambda_i)}{x - \lambda_i} \right] + \sum_{a\in \partial A} \frac{\partial_a \ln Z^A_N}{x - a} = 0$$

which can be rewritten:

$$W_2(x, x) + (W_1(x))^2 + (1 - 2/\beta)W'_1(x) - \int_{A} \frac{d\xi}{2i\pi} \frac{V'(\xi)W_1(\xi)}{x - \xi} + \sum_{a\in \partial A} \frac{\partial_a \ln Z^A_N}{x - a} = 0$$

- For any $n \geq 1$, there is a quadratic relation between $W_{n+1}, W_n, \ldots, W_1$
A priori control on correlators

For the conditioned measure $\mu_{N}^{A}$

consider $N, (N_h)_h \to \infty$ with $\epsilon_h = N_h/N$ fixed, close enough to $\epsilon_h^*$

- There is an equilibrium measure $\mu_{eq}^\epsilon$ (depending smoothly on $\epsilon$)

So: \[
N^{-1}W_1(x) \xrightarrow{N\to\infty} \int \frac{d\mu_{eq}(\xi)}{x - \xi}
\]

- From global large deviations:

\[
\left| W_1(x) - N \int \frac{d\mu_{eq}^\epsilon(\xi)}{x - \xi} \right| \leq c_1 [d(x, A)] (N \ln N)^{1/2}
\]

\[
|W_m(x_1, \ldots, x_m)| \leq \left( \prod_{i=1}^{m} c_m[d(x_i, A)] \right) (N \ln N)^{m/2}
\]
Rigidity of the Schwinger-Dyson equations

By recursive analysis of the Schwinger-Dyson equation:

\[
\left| W_1(x) - N \int \frac{d\mu_{eq}(\xi)}{x-\xi} \right| \leq c_1 [d(x, A)] (N \ln N)^{1/2}
\]

\[
|W_m(x_1, \ldots, x_m)| \leq \left( \prod_{i=1}^m c_m [d(x_i, A)] \right) (N \ln N)^{m/2}
\]

\[
\downarrow \text{thanks to off-criticality}
\]

\[
(W_1(x) - N \int \frac{d\mu_{eq}(\xi)}{x-\xi}) \rightarrow W_1^{[0]}(x)
\]

\[
|W_m(x_1, \ldots, x_m)| \leq \left( \prod_{i=1}^m c'_m [d(x_i, A)] \right) N^{2-m}
\]

\[
\downarrow
\]

\[
W_m(x_1, \ldots, x_m) = \sum_{k \geq m-2} N^{-k} W_m^{[k]}(x_1, \ldots, x_m) + O(N^{-K})
\]

for all K (no uniformity)
Back to the partition function

\[
\frac{Z_{N}^{A;V_{1}}}{Z_{N}^{A;V_{0}}} = \exp \left[ - \left( \frac{\beta}{2} \right) \sum_{k \geq -2} N^{-k} \int_{A} \frac{dx}{2i\pi} \partial_{t}V_{t}(x) W_{1}^{V_{t};[k+1]}(x) + O(N^{-(K-1)}) \right]
\]

To deduce an expansion for \( Z_{N}^{A;V} \), we need

- \( V_{0} \) such that \( Z_{N}^{A;V_{0}} \) is exactly known
- an interpolation \( (V_{t})_{t \in [0,1]} \) from \( V_{t=1} = V \), staying uniformly \((g + 1)\)-cuts and off-critical

Idea: interpolate in the space of equilibrium measures

\[
(\mu_{eq}^{t})_{t \in [0,1]} \quad \text{↔} \quad (V_{t})_{t \in [0,1]}
\]

\[
\int_{A} \ln |x - y| d\mu_{eq}^{t}(y) - V_{t}(x) = C_{t} \quad \text{with equality } \mu_{eq}^{t}\text{-everywhere}
\]
An interpolation path ...

\[ Z^A_N[V_t] \sim \prod_{0 \leq h < h' \leq g} \left| \frac{a_h + b_h - a_{h'} - b_{h'}}{2} \right|^N e^{\frac{\epsilon_h \epsilon_{h'}}{\beta}} \prod_{h=0}^g \left( \text{Selberg } \beta \text{–Gaussian integral over } \mathbb{R}^{N_h} \right) \]
Sums and interferences - 1/3

We initially wanted to compute

\[ Z_N^A = \sum_{N_0 + \ldots + N_g = N} \frac{N!}{\prod_{h=0}^g N_h!} Z_{(N_0, \ldots, N_g)}^{(A_0, \ldots, A_g)} \]

- From global large deviations:

\[ Z_N^A = \left( \sum_{|N - Ne^*| \leq \ln N} \frac{N!}{\prod_{h=0}^g N_h!} Z_N^A \right) (1 + O(e^{-cN})) \]

- For \( N - Ne^* \in o(N) \), we just proved, with \( \epsilon = (N_h/N)_{1 \leq h \leq g} \)

\[ \frac{N!}{\prod_{h=0}^g N_h!} Z_N^A = N^\gamma N^{+\gamma'} \exp \left[ \sum_{k \geq -2} N^{-k} F_k(\epsilon) + O(N^{-K}) \right] \]

- Extra lemma: \( F_k(\epsilon) \) are smooth functions of \( \epsilon \approx \epsilon^* \)

\[ F'_{-2}(\epsilon^*) = 0 \quad \text{and} \quad F''_{-2}(\epsilon^*) < 0 \]
We plug the asymptotic formula and use a Taylor expansion at $\epsilon \approx \epsilon^*$

- E.g. up to $o(1)$:

$$Z^A_N = N\gamma N + \gamma' e^{N^2 F_{-2}(\epsilon^*)} + N F_{-1}(\epsilon^*) + F_0(\epsilon^*)$$

$$\times \left( \sum_{|N - N\epsilon^*| \leq \ln N} e^{\frac{1}{2} F_{-2}''(\epsilon^*) \cdot (N - N\epsilon^*) \otimes 2 + F_{-1}'(\epsilon^*) \cdot (N - N\epsilon^*)} \right) \left( 1 + O(e^{-c(\ln N)^3/N}) \right)$$

It is the general term of a super-exponentially fast converging series:

$$Z^A_N = N\gamma N + \gamma' e^{N^2 F_{-2}(\epsilon^*)} + N F_{-1}(\epsilon^*) + F_0(\epsilon^*)$$

$$\times \left( \sum_{N \in \mathbb{Z}^g} e^{\frac{1}{2} F_{-2}''(\epsilon^*) \cdot (N - N\epsilon^*) \otimes 2 + F_{-1}'(\epsilon^*) \cdot (N - N\epsilon^*)} \right) \left( 1 + O(e^{-c(\ln N)^2}) \right)$$

- We recognize $\Theta_{-N\epsilon^*}(F_{-1}'|F_{-2}'')$
Including higher orders yields terms of the form

$$\sum_{N \in \mathbb{Z}^g} \left( \frac{1}{r!} \prod_{i=1}^{r} \frac{F_{k_i}^{(l_i)}(\epsilon^*) \cdot (N - N\epsilon^*) \otimes l_i}{l_i!} \right) e^{\frac{1}{2} Q \cdot (N - N\epsilon^*) \otimes^2 + w \cdot (N - N\epsilon^*)}$$

We recognize

$$\left( \frac{1}{r!} \prod_{i=1}^{r} \frac{F_{k_i}^{(l_i)}(\epsilon^*) \cdot \nabla_{\mathbf{w}} \otimes l_i}{l_i!} \right) \Theta_{-N\epsilon^*}(\mathbf{w}|Q)$$

Here $Q = F_{-2}(\epsilon^*)$ and $\mathbf{w} = F_{-1}(\epsilon^*)$

We justified step by step the heuristics of  

Bonnet, David, Eynard '00, Eynard '07
Oscillatory asymptotic expansion

\[ Z^A_N = N^{\gamma_N + \gamma'} (D_N \Theta_{-N\epsilon_x}) (F'_{-1} | F''_{-2}) \exp \left[ \sum_{k \geq -2} N^{-k} F_k + O(N^{-\infty}) \right] \]

where \[ D_N = \sum_{r \geq 0} \frac{1}{r!} \sum_{\ell_1, \ldots, \ell_r \geq 1} N^{-(\sum_i k_i + \ell_i)} \prod_{i=1}^r \frac{F_k^{(\ell_i)} \cdot \nabla_{\ell_i}}{\ell_i!} \]

acts as a differential operator on the Siegel theta function

\[ \Theta_\mu(w|Q) = \sum_{m \in \mathbb{Z}^g} e^{w \cdot (m+\mu) + \frac{1}{2} (m+\mu) \cdot Q \cdot (m+\mu)} \]

Moving characteristics

\[ \mu = -N\epsilon_x \mod \mathbb{Z}^g \]

Quadratic form

\[ Q = F''_{-2} = 2i\pi (\beta/2) \times (\text{period matrix}) < 0 \]
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Generalization ...

... to real-analytic k-body interactions

$$d\mu_N^A = \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \exp \left( \sum_{k=1}^{r} \frac{N^{2-k}}{k!} \sum_{i_1, \ldots, i_k = 1}^{N} T_k(\lambda_{i_1}, \ldots, \lambda_{i_k}) \right) \prod_{i=1}^{N} d\lambda_i$$

- Equilibrium measure & local large deviations provided

$$\mathcal{E}[\mu] = \frac{\beta}{2} \iint \ln |x_1 - x_2| d\mu(x_1) d\mu(x_2) + \sum_{k=1}^{r} \frac{1}{k!} \int T_k(x_1, \ldots, x_k) \prod_{i=1}^{k} d\mu(x_i)$$

has a unique minimum

- Global large deviations provided $\mathcal{E}''[\mu_{eq}] < 0$

- Similar asymptotic results

- Coefs. of expansions are given by a “blobbed” topological recursion

Borot ’13
General ideas

- Nature of expansion depends on the topology of the spectrum
  
  connected  $\rightarrow$  1/N expansion
  
  gaps  $\rightarrow$  ... + interference patterns

- Structure of expansion is influenced by singularities of the measure on the “moduli space” \( \mathcal{M} = A^N / \mathcal{G}_N \)

\[
\prod_{i<j} |\lambda_j - \lambda_i|^\beta = \text{non-analyticity on } \partial \mathcal{M}
\]
Open problems

- Singular V’s and uniform asymptotics around critical points
  - asymptotics of (skew) orthogonal polynomials in the bulk
  - universality and computing tails of universal laws

- Complex-valued V
  - Berry-Esséen type estimates in CLT

- Same questions for $\lambda_i \in \mathbb{C}$

- Same questions for multi-matrix models
  - asymptotics of biorthogonal polynomials