The Open Quantum Brownian Motion: a Random Walk on Quantum Noise.
Two facets of Quantum Noise:

— Observing, manipulating, controlling quantum systems. These are probabilistic by nature, and if time evolution is taken care of these are stochastic processes.

— Extending applications of probability theory by incorporating quantum effects, e.g. quantum stochastic processes (as open quantum systems).

Aim:
Use the Open Quantum Brownian Motion (the definition will come) to illustrate a few notions/properties of Quantum Noise.

— Progressive collapse in non-demolition quantum measurements (as an illustration of the first point above)

— Transition in quantum random walks from diffusive to ballistic regimes. (as an illustration of the second point above)
I) Progressive collapse in non-demolition measurements

Progressive field-state collapse and quantum non-demolition photon counting

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How to record the cavity states? Is it faithful representation?

Why does the p.d.f. changes with time?

How does it evolve? What’s the dynamics? Why does it become peaked (collapsed)?

Use (quantum) random walks to illustrate the universality of this phenomenon.
II) Quantum trajectories and a diffusive to ballistic transition in open quantum Brownian motion.

— Open Quantum Brownian Motion:
A quantum random motions on the line for a « quantum walker » with internal and orbital degrees of freedom.

A Brownian like regime.

A ballistic like regime.

Transition in the behavior of the internal « gyroscope »:
from oscillation to bi-stability with jumps (analogous to Bohr quantum jumps)

— The ballistic flips are not due to collision but to the flips of the internal gyroscope.
Outline:

— Quantum parallelism on (classical) random walks and measurements.
  (progressive collapse or `quantum computation’ with random walks)
— A glance at Quantum Noise and quantum trajectories.
  (how analogue to classical processes is it?)
— Open quantum random walk trajectories.
  (a random walk but with an internal quantum gyroscope)
— Two regimes and bi-stability (diffusion or ballistics).
  (gyroscope quantum jumps and flips a la Kramer’s)
— How to make OQBM conformally invariant?
  (if time permits….)
Quantum Parallelism on Random Walks.

— Consider a random walk on the line:

\[ \omega = (+, +, -, +, \cdots) = (\epsilon_1, \epsilon_2, \epsilon_2, \cdots) \]

Events:

\[ X_n(\omega) = X_0 + \epsilon_1 + \cdots + \epsilon_n \]

— We can use Quantum Mechanics to describe it, keeping all possible walks at once:

Let \( \mathbb{C}^\mathbb{Z} \) be the walker Hilbert space, with basis \( |x\rangle, x \in \mathbb{Z}. \)

Let \( \mathbb{C}^2 \) be an auxiliary “coin” Hilbert space, with basis \( |\pm\rangle. \)

\[
|\psi_n\rangle := \frac{1}{2^{n/2}} \sum_{[\omega_n]} |X_n(\omega)\rangle \otimes |[\omega_n]\rangle
\]

with a sum over all walks of \( n \) steps. \( |[\omega_n]\rangle = |\epsilon_1\rangle \otimes \cdots \otimes |\epsilon_n\rangle \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \)

— There is no more information than in the usual description, all walks appear in the sum with equal weight:

simple “book keeping” or “quantum parallelism”, keeping track of the position “\( X \)”, of the history “\( w \)”. 
Quantum Parallelism as dynamical process.

— We view the auxiliary Hilbert space of an auxiliary coin/probe. We consider a collection of probes. Assume that each probes are all initially prepared in a state
\[ |\phi\rangle \in \mathbb{C}^2 \]

— Consider the elementary (unitary) evolution (walker+one probe):
\[
|x\rangle \otimes |\phi\rangle \rightarrow \frac{1}{\sqrt{2}} |x + 1\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}} |x - 1\rangle \otimes |-\rangle
\]

After e.g. there iterations:
\[
2^{-3/2} \left( |x_0 + 3\rangle \otimes |+++\rangle + |x_0 + 1\rangle \otimes (|++--\rangle + |+-+-\rangle + |-++\rangle) + |x_0 - 1\rangle \otimes (|--+-\rangle + |-++\rangle + |++--\rangle) + |x_0 - 3\rangle \otimes |----\rangle \right)
\]

— The occurrence probability of a given walk is the modulus square of the amplitude (« quantum measurement »). A given output measurement projects on a given realization of the random walk.
Quantum noise and repeated quantum interaction.

\( S := \text{Quantum System} \)

- Hilbert space:
  \( \mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \cdots \)

- Algebras of observable:
  \( \mathcal{B}_n := \mathcal{A}_s \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \otimes \mathbb{I} \)
  \( \mathcal{A}_s \subset \mathcal{B}_n \subset \mathcal{B}_m \), for \( n < m \)

- **Gain of information**: by testing output observable on the n-th first probes, but a probabilistic gain because of Q.M.
  
  --- Notion of noise (with a measure, the probe state).
  --- Notion of arrow of time (information on the probes)
  --- Reservoir, by summing over the probes.

- Measurement of some observables on the output probes (not on the Q-system):
  The quantum filtration is reduced to a classical filtration.
  
  Quantum trajectories = (classical random process, with events the output measurements)
Random walk with tilted measurements.

— Once the auxiliary space is viewed as that of probes we may imagine measuring the « spin » probes in another direction, say at an angle « theta ».

\[ |\pm^u\rangle = \cos \vartheta /2 |\pm\rangle \pm \sin \vartheta /2 |\mp\rangle \]

Measuring the probes (successively) given a random output +/-, (with probability given by the (square of the) projection of the state on the above vectors). Hence the state of the walker after the probe measurement is random.

— What is this random state evolution?

At time t:

\[ |\psi_t\rangle = \sum_x \psi_{x,t} |x\rangle, \text{ with } \sum_x |\psi_{x,t}\rangle^2 = 1 \]

Elementary evolution:

\[ |\psi_t\rangle \rightarrow \frac{1}{\sqrt{2}} \sum_x \psi_{x,t} (|x+1\rangle \otimes |+\rangle + |x-1\rangle \otimes |--\rangle). \]

Measuring the spin in the « u » direction, we have to project the probe on the above states, and the probability of +/- is the modulus square of the amplitude:

Probe measurement:

\[ |\psi_{t+1}\rangle = \frac{1}{\sqrt{2}p_{\pm}(t)} \sum_x \left( \psi_{x-1,t} \langle \pm^u |+\rangle + \psi_{x+1,t} \langle \pm^u |--\rangle \right) |x\rangle \]

with probability:

\[ p_{\pm}(t) := \frac{1}{2} \sum_x \left| \psi_{x-1,t} \langle \pm^u |+\rangle + \psi_{x+1,t} \langle \pm^u |--\rangle \right|^2. \]

— The probability distribution of the walker position is then always delocalized. « But » this distribution in momentum space collapses to a peaked measure !! i.e. the Fourier component of the wave function collapse almost surely.
— In momentum space  \[ \psi_{x,t} = \frac{1}{\sqrt{N}} \sum_k \phi_{k,t} e^{2i\pi k x/N}, \text{ with } \sum_k |\phi_{k,t}|^2 = 1 \]

for a walk on \( \mathbb{Z}_N \)

— The momentum distribution \( |\phi_{k,t}|^2 \) are random but:

Claim:

\[ \lim_{t \to \infty} |\phi_{k,t}|^2 = \delta_{k,k_\infty}, \text{ with } k_\infty \text{ random} \]

\[ \mathbb{P}[k_\infty = p] = |\phi_{p,t=0}|^2 \]

— Open Quantum Random Walks but wit tilted probe measurements define « progressive » quantum measurements of the momentum.

Progressive: the collapse/measurement takes some time, but the collapse is exponential in time.
Proof: 
— The random state evolution in momentum space
\[ |\phi_{k,t+1}|^2 = \frac{1}{2p_{\pm}(t)} \left[ 1 \pm \sin \vartheta \cos \left( 4\pi k/N \right) \right] |\phi_{k,t}|^2 \]
\[ p_{\pm}(t) = \frac{1}{2} \sum_l \left( 1 \pm \sin \vartheta \left[ \cos(4\pi l/N) |\phi_{l,t}|^2 \right] \right). \]

preserving the normalisation of the momentum distribution.

— Why does the momentum distribution converges and collapse?

\[ t \to |\phi_{k,t}|^2 \] are bounded martingales.
as such they converge almost surely and in L1

A martingale is a random process which is conserved
in mean when conditioned on the past:
\[ \mathbb{E}[|\phi_{p,t+1}|^2 | \mathcal{F}_t] = |\phi_{p,t}|^2 \]
And here it is tautologically true.

— Bayesian interpretation.
The above formula is actually the formula for conditional probability:
\[ |\phi_{k,t+1}|^2 = \frac{p(\pm |k| |\phi_{k,t}|^2}{p_{\pm}(t)}, \text{ with } p(\pm |k| := \frac{1}{2} \left[ 1 \pm \sin \theta \cos(4\pi k/N) \right] \]

Quantum mechanics code for the Bayesian rules.
Collapse is « classical »: An echo of random Bayesian updating
Generalisation and application:

— The previous statement is general:

Iterating ad infinitum « system-probe interaction » plus « probe measurement » results in the collapse of the system probability distribution, provided that the interaction between the probes and the system preserves a specified basis of states, call « pointer » states.

— These are mesoscopic (quantum non-demolition) measurements (they become macroscopic only when an infinite number of probes have been involved) A Bayesian point of view.

— Many applications: Measurement continuously in time, (e.g. imaging « Bohr » virtual quantum jumps (experiments/theory)). Quantum control, etc…. 

These are identical phenomena.
Open quantum random walks (I)

— Back to the quantum parallelism on random walks but we add internal degree of freedom, « quantum gyroscope » to the walker.

* Hilbert space of states:
\[ \mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p^\otimes \]
\[ \mathcal{H}_c = \mathbb{C}^2, \quad \mathcal{H}_o = \mathbb{C}^\mathbb{Z} \]
\[ \mathcal{H}_p^\otimes = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \]

Coins are reset at each step.

— Interaction (without measurements):
On state \(|\psi\rangle_c \otimes |n\rangle_o \otimes |\phi\rangle_p\) in \(\mathcal{H}_c \otimes \mathcal{H}_o \otimes \mathcal{H}_p\), the unitary evolution

\[
(B_+ |\psi_c\rangle \otimes |n + 1\rangle_o \otimes |+\rangle_p + (B_- |\psi_c\rangle \otimes |n - 1\rangle_o \otimes |\rangle_p

Iterating interactions with different probes produce an «entangled» state, sum of states each indexed by a random walk (as before).

— The original definition of open QRW was the mean of this process (not keeping track of the measurements).

Open quantum random walks trajectories (II).

— Measurements and «quantum trajectories»:

\[
(B_+|\psi_c\rangle \otimes |n+1\rangle_o \otimes |+\rangle_p + (B_-|\psi_c\rangle \otimes |n-1\rangle_o \otimes |-\rangle_p
\]

Measuring the probes, one may find + or - with probabilities, and then «project» the state on |+\rangle or |->:

\[
\propto (B_\pm|\psi_c\rangle \otimes |n\pm1\rangle_o \text{ with probability } c \langle\psi|B_\pm^\dagger B_\pm|\psi_c\rangle
\]

The events, the output of the probe measurements, are in one-to-one correspondence with random walks.

— These are classical random processes with values on the line x the internal states.

— For mixed (internal) state: \text{ie. the system is not described by a vector but by a density matrix (a positive hermitian normalised matrix).}

If after n step,

\[
(\rho_n, \ x_n)
\]

The updating is:

\[
(p_\pm(n)^{-1} B_\pm \rho_n B_\pm^\dagger, \ x_n \pm 1)
\]

with probability:

\[
p_\pm(n) = \text{tr}(B_\pm \rho_n B_\pm^\dagger)
\]

For a pure state:

\[
\rho_n = |\psi_n\rangle \langle \psi_n|
\]

The matrices B are the moduli parameters of the walks. Events are random walks (output measurements), but with probabilities induced by the gyroscope motion.
Numerical simulations:

— Unitarity or normalisation of probabilities imposes: \[ B_+^\dagger B_+ + B_-^\dagger B_- = \mathbb{I} \]

A choice for the simulations:

\[ B_+ = \delta^{-1} \left( \begin{array}{c} u \\ r \\ s \\ v \end{array} \right) \quad \text{and} \quad B_- = \delta^{-1} \left( \begin{array}{c} -v \\ r \\ s \\ -u \end{array} \right) \]

with \( \delta = \sqrt{u^2 + v^2 + r^2 + s^2} \)

A Brownian like regime.

\( u = 1.005, \, v = 1.00 \) and \( r = -s = 0.00015 \)

A ballistic like (but diffusive) regime.

\( u = 1.1, \, v = 1.00 \) and \( r = -s = 0.00015 \)

— Kramer’s like transition for a particle in a double well potential (... but not quite).
Scaling limit: «Open quantum Brownian motion».

— In the scaling limit one gets a time continuous process (with continuous measurement)

The scaling limit is $\epsilon \to 0$, $t = n\epsilon$ fixed, and $dx^2 \sim dt$.

— As for classical Brownian motion, scale time, distance and moduli simultaneously:

$$B_\pm = \frac{1}{\sqrt{2}}[\mathbb{I} \pm \sqrt{\epsilon}N + \epsilon(iH_\pm \pm M - \frac{1}{2}N^\dagger N) + o(\epsilon)]$$

with $\epsilon$ a small parameter and $H_\pm, M$ hermitian but not $N$.

— In the scaling limit, the series of +/- of output probe measurements
become a random process driven by a « classical » Brownian motion.

$$d\rho_t = (i[H, \rho_t] + L_N(\rho_t)) dt + D_N(\rho_t) dB_t, \quad dX_t = U_N(\rho_t) dt + dB_t,$$

with $L_N(\rho) := N\rho N^\dagger - \frac{1}{2}(N^\dagger N\rho + \rho N^\dagger N)$

$$D_N(\rho) := N\rho + \rho N^\dagger - \rho U_N(\rho) \text{ and } U_N(\rho_t) := \text{tr}(N\rho + \rho N^\dagger)$$

— Both the density matrix and the position have stochastic evolution
but the position drift is guided by the internal gyroscope.
Transition between two regimes:

Take $H := \omega_0 \sigma^2$ and $N = a \sigma^3$ with $\sigma^{1,2,3}$ the usual Pauli matrices.

Describe $\rho_t$ as a pure state: $\rho_t = \frac{1}{2}(I + q_1 \sigma^1 + q_3 \sigma^3)$ with $q_1 = \sin \theta$, $q_3 = \cos \theta$.

Eqs. $d\rho_t = (i[H, \rho_t] + L_N(\rho_t)) dt + D_N(\rho_t) dB_t$, becomes:

$$d\theta_t = -2(\omega_0 + a^2 \sin \theta_t \cos \theta_t) dt - 2a \sin \theta_t dB_t$$

— Rabi like oscillations:

$a^2 < \omega_0$

— Quantum jumps

$a^2 > \omega_0$

— Two potential minima, with Kramer’s transition between them.

« but not quite », because jumps arise for « strong noise ».
Ballistically induced diffusion:

— Trajectories are ballistic, with seesaw profiles induced by gyroscope flips, and large mean free paths.

\[
dX_t = U_N(\rho_t) \, dt + dB_t \quad \text{becomes:}
\]

\[
dX_t = 2a \cos \theta_t \, dt + dB_t
\]

with \( 2a \cos \theta^*_\pm \approx \mp 2a \)

The seesaw behavior is not due to collision but to the internal gyroscope behavior.

— But at very large time the position is Gaussian, with large effective diffusion constant.

\[
\mathbb{E}[X_t^2] = D_{\text{eff}} \, t \quad \text{with} \quad D_{\text{eff}} = 1 + 4a^4/\omega_0^2
\]

Question: Can we find similar phenomena (phenomenological description) producing a large effective diffusion constant (at large time)?
How to make the Open Quantum Brownian Motion conformally invariant?

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- The 2D Brownian motion is conformally invariant (Levy), up to random time parametrization:
  \[ dB_t \, dB_t = 2dt \]

- A way to render a 2D quantum walker conformally invariant is to couple it to an internal conformal field theory (CFT).
  Which CFT? Which evolution equation for the internal state, The walker position will satisfy a SDE with a drift of the form:
  \[ dX_t = U_N(\rho_t) \, dt + dB_t \]

- To preserve conformal the drift has to be a (0,1)-form, because time scales under conformal transformation:
  \[ dZ_t = \bar{\Omega}^D(Z_t, \bar{Z}_t) \, dt + dB_t. \]
  Hence, the CFT has to have a U(1) current and
  \[ \bar{\Omega}^D(Z_t, \bar{Z}_t) := 4\sqrt{\kappa} \omega^D_t \langle \bar{J}(\bar{Z}_t) \rangle \]

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- We then have to write the corresponding SDE for the evolution of the internal CFT state \( \omega^D_t \langle \ldots \rangle \) according to the rule of the OQBM.

\[
\begin{align*}
\frac{d\omega^D_t \langle A \rangle}{dt} &= \sqrt{\kappa} \left[ \omega^D_t \langle J(Z_t) A + AJ(Z_t) \rangle - 2 \omega^D_t \langle J(Z_t) \rangle \omega^D_t \langle A \rangle \right] dB_t + \text{c.c.}, \\
\frac{dZ_t}{dt} &= 4\sqrt{\kappa} \omega^D_t \langle \bar{J}(\bar{Z}_t) \rangle dt + dB_t
\end{align*}
\]

The SDE for the internal state has an additional drift term, \( \kappa \omega^D_t \langle L^*_t[A] \rangle dt \) but in general it does not contribute.
Thank you.
Open Quantum Brownian motion:

--- Quantum trajectories: Probes are measured (with random outputs), and the system density matrix evolves randomly (according to the measurement outputs).

more below..... .... and this can be generalised with many packets and/or entangled states.

--- Quantum dynamical map: Probes (reservoir) are *not* measured but trace out, and the system density matrix evolves with Lindblad equation

\[
\partial_t \tilde{\rho}_t = -\frac{1}{2} [P, [P, \tilde{\rho}_t]] + i \left( N[P, \tilde{\rho}_t] + [P, \tilde{\rho}_t]N^\dagger \right) + i [H, \tilde{\rho}_t] + L_N(\tilde{\rho}_t)
\]

for the density matrix \( \tilde{\rho}_t := \int dx \rho(x, t) \otimes |x\rangle \langle x| \)

--- Quantum Stochastic equation: Probes (reservoir) are *not* measured but *not* trace out, and the *total* state evolves with a quantum-SDE.

\[
dA_t = i[P + iN^\dagger, A_t] d\xi_t + i[P - iN, A_t] d\xi_t^\dagger + L_*(A_t) dt
\]

for observable A, and dual Linbladian L., and (quantum) noises: \( d\xi_t d\xi_t^\dagger = dt, \quad d\xi_t^\dagger d\xi_t = 0 \)