Cylindrical contact homology in dimension 3 via intersection theory and more

Jo Nelson

the IAS and Columbia University

Short Talks, October 1, 2014
What is a contact manifold?

A contact structure $\xi$ on $M^{2n-1}$ is a maximally non-integrable hyperplane distribution...
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$\iff$

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Here: $\alpha = dz - ydx$
Choose a contact form $\alpha$.

**Definition**

The Reeb vector field $R_\alpha$ is uniquely determined by

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, \cdot) = 0$. 

Reeb orbits are Hopf fibers of $S^3$. 

Patrick Massot

http://www.nilesjohnson.net/hopf.html

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Reeb flow

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Cylindrical contact homology in dimension 3
A dream for a chain complex

Assume: $M$ closed and $\alpha$ nondegenerate
A dream for a chain complex

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“Do” Morse theory on

$$A : C^\infty(S^1, M) \to \mathbb{R},$$

$$\gamma \mapsto \int_\gamma \alpha.$$
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$$\mathcal{A} : \mathcal{C}^\infty(S^1, M) \rightarrow \mathbb{R},$$

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Proposition

$$\gamma \in \text{Crit}(\mathcal{A}) \iff \gamma \text{ is a closed Reeb orbit}.$$
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“Do” Morse theory on $A$:

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**Proposition**

$$\gamma \in \text{Crit}(A) \iff \gamma \text{ is a closed Reeb orbit}.$$ 

- Grading on orbits given by Conley-Zehnder index,
- $C_*(\alpha) = \{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}$
Gradient flow lines no go; use finite energy pseudoholomorphic cylinders $u \in \mathcal{M}(\gamma_+; \gamma_-)$, where $\gamma_\pm$ are Reeb orbits of periods $T_\pm$. 

Hope this is independent of our choices.

Conjecture (Eliashberg-Givental-Hofer '00)
Assume a minimal amount of things. Then $(\mathcal{C}_\bullet(\alpha), \partial)$ forms a chain complex and $H(\mathcal{C}_\bullet(\alpha), \partial)$ is independent of $\alpha$ and $\tilde{J}$.
A dream...

Gradient flow lines no go; use finite energy pseudoholomorphic cylinders $u \in \mathcal{M}(\gamma_+; \gamma_-)$, where $\gamma_\pm$ are Reeb orbits of periods $T_\pm$.

\[
\begin{aligned}
    u &:= (a, f) : (\mathbb{R} \times S^1, j) \rightarrow (\mathbb{R} \times M, \tilde{J}) \\
    \bar{\partial}_j \tilde{J} u &:= du + \tilde{J} \circ du \circ j \equiv 0
\end{aligned}
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    u := (a, f) : (\mathbb{R} \times S^1, j) \rightarrow (\mathbb{R} \times M, \tilde{J}) \quad \lim_{s \rightarrow \pm \infty} a(s, t) = \pm \infty
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    \tilde{\partial}_j, \tilde{J} u := du + \tilde{J} \circ du \circ j \equiv 0 \quad \lim_{s \rightarrow \pm \infty} f(s, t) = \gamma_\pm(T_\pm t)
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up to reparametrization.
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$\partial : C_* \to C_{*-1}$ is a weighted count of pseudoholomorphic cylinders up to reparametrization.
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**Conjeorem (Eliashberg-Givental-Hofer '00)**

*Assume a minimal amount of things. Then \((C_* (\alpha), \partial))\) forms a chain complex and \( H(C_* (\alpha), \partial) \) is independent of \( \alpha \) and \( \tilde{J} \).*
The nightmare of contact homology

- Transversality for multiply covered curves...good luck
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Desired compactification

Adding to 2 becomes hard
Hope: the big reveal

- Automatic transversality results of Wendl, Hutchings, and Taubes in **dimension 3**.
Automatic transversality results of Wendl, Hutchings, and Taubes in \textbf{dimension 3}.

Understand basic arithmetic and the Conley-Zehnder index.

Definition

Assume $c_1(\xi) = 0$. For today restrict to when $R_\alpha$ has only contractible orbits.

We say a contact form is dynamically separated whenever

(i) All closed simple contractible Reeb orbits $\gamma$ satisfy $3 \leq \mu_{CZ}(\gamma) \leq 5$.

(ii) $\mu_{CZ}(\gamma_k) = \mu_{CZ}(\gamma_{k-1}) + 4$, $\gamma_k$ is the $k$-th iterate of a simple orbit $\gamma$.

Theorem (N.) \[ \partial^2 = 0, \] invariance under choice of $\tilde{J}$ and dynamically separated $\alpha$. 

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Cylindrical contact homology in dimension 3
Hope: the big reveal

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- Understand basic arithmetic and the Conley-Zehnder index
- Realize your original thesis project contained a useful geometric perturbation
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**Definition**

Assume $c_1(ξ) = 0$. For today restrict to when $R_α$ has only contractible orbits. We say a contact form is **dynamically separated** whenever

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**Theorem (N.)**

$\partial^2 = 0$, *invariance under choice of $\tilde{J}$ and dynamically separated $\alpha$.*
A better reveal

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A better reveal

Do more index calculations
Learn some intersection theory
Team up with Hutchings

Remaining obstruction to $\partial^2 = 0$ can be excluded!

Definition

A nondegenerate $(M^3, \xi = \ker \alpha)$ is dynamically convex whenever $c_1(\xi) |_{\pi_2(M)} = 0$ and every contractible $\gamma$ satisfies $\mu_{CZ}(\gamma) \geq 3$.

Any convex hypersurface transverse to the radial vector field $Y$ in $(\mathbb{R}^4, \omega_0)$ admits a dynamically convex contact form $\alpha := \omega_0(Y, \cdot)$.

Theorem (Hutchings-N.)

If $(M^3, \alpha)$ is dynamically convex and every contractible Reeb orbit $\gamma$ has $\mu_{CZ}(\gamma) = 3$ only if $\gamma$ is simple then $\partial^2 = 0$.
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*If $(M^3, \alpha)$ is dynamically convex and every contractible Reeb orbit $\gamma$ has $\mu_{CZ}(\gamma) = 3$ only if $\gamma$ is simple then $\partial^2 = 0$.***
Too legit to quit

Still stuck on Invariance....

Throw in the entire kitchen sink

Non-equivariant formulations,
domain dependent almost
complex structures,
obstruction bundle gluing

Family Floer homology constructions to get an
$S^1$-equivariant

theory which should be
$SH_{S^1}^*$

Tensor with $Q$ to get back
$CH^*$

Theorem (Hutchings-N; in progress)

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**Theorem (Hutchings-N; in progress)**

*INVARIANCE! Obtained for dynamically convex $(M^3, \alpha)$ wherein a contractible $\gamma$ has $\mu_{CZ}(\gamma) = 3$ only if $\gamma$ is simple.*
Commutations for Seifert fiber spaces
Onwards

- Computations for Seifert fiber spaces
- Connections to Chen-Ruan orbifold homology and string topology
Onwards

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- Look at dimensions $> 3$??
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- Computations for Seifert fiber spaces
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- Look at dimensions $> 3$??
- Other dynamical questions involving contact structures
The end!

Thanks!

E(u) := sup \int_{\Sigma} u^* \omega_{\phi}.