Cylindrical contact homology as a well-defined homology?

Jo Nelson

Columbia University and the IAS

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What is a contact manifold?

A contact structure $\xi$ on $M^{2n-1}$ is a maximally non-integrable hyperplane distribution...
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If $\alpha$ is a 1-form on $M$ and

- $\alpha \wedge (d\alpha)^{n-1}$ is a volume form
- $\Leftrightarrow d\alpha|_\xi$ is nondegenerate

then $\xi := \ker \alpha$ is a contact structure.
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Above:

\[ \alpha = dz - ydx \]
Choose a contact form $\alpha$.

**Definition**

The Reeb vector field $R_\alpha$ is uniquely determined by

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, \cdot) = 0$. 

Reeb orbits are Hopf fibers of $S^3$, $\alpha_0 = i^2 (ud\bar{u} - \bar{u} du + vd\bar{v} - \bar{v} dv)$.
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A dream for a chain complex

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“Do” Morse theory on

$$\mathcal{A} : C^\infty(S^1, M) \to \mathbb{R},$$

$$\gamma \mapsto \int_{\gamma} \alpha.$$
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Proposition

$$\gamma \in \text{Crit}(A) \iff \gamma \text{ is a closed Reeb orbit}.$$
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- Grading on orbits given by Conley-Zehnder index,
- $C_*(\alpha) = \{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}$
Gradient flow lines no go; use finite energy pseudoholomorphic cylinders $u \in M(\gamma_+; \gamma_-)$, where $\gamma_\pm$ are Reeb orbits of periods $T_\pm$. 
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$$u := (a, f) : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, \tilde{J})$$

$$\bar{\partial}_{j, \tilde{J}} u := du + \tilde{J} \circ du \circ j \equiv 0$$
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\( \bar{\partial} : C_* \to C_{*-1} \) is a weighted count of pseudoholomorphic cylinders up to reparametrization.
A dream...

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- $\partial : C_* \to C_{*-1}$ is a weighted count of pseudoholomorphic cylinders up to reparametrization
- Hope this is independent of our choices.

**Conjeorem (Eliashberg-Givental-Hofer '00)**

Assume a minimal amount of things. Then $(C_*(\alpha), \partial))$ forms a chain complex and $H(C_*(\alpha), \partial)$ is independent of $\alpha$ and $\tilde{J}$.
The nightmare of contact homology

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Desired compactification

Adding to 2 becomes hard
Automatic transversality results of Wendl, Hutchings, and Taubes in \textbf{dimension 3}.
Hope

- Automatic transversality results of Wendl, Hutchings, and Taubes in **dimension 3**.
- Understand basic arithmetic and the Conley-Zehnder index.
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Realize your original thesis project contained a useful geometric perturbation.
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\begin{align*}
\text{Assume } c_1(\xi) &= 0. \\
\text{For today restrict to when } R^\alpha \text{ has only contractible orbits.}
\end{align*}
\]

We say a contact form is dynamically separated whenever the following hold:

(i) All closed simple contractible Reeb orbits $\gamma$ satisfy $3 \leq \mu_{CZ}(\gamma) \leq 5$.

(ii) $\mu_{CZ}(\gamma_k) = \mu_{CZ}(\gamma_{k-1}) + 4$, $\gamma_k$ is the $k$-th iterate of a simple orbit $\gamma$.
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Cylindrical contact homology as a well-defined homology?
Simple singularities appear as origin of $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU_2(\mathbb{C})$. The origin is an isolated quotient singularity. The variety $\mathbb{C}^2/\Gamma$ can be identified with the hypersurface $\{f - 1\}_\Gamma(0) \subset \mathbb{C}^3$. The link is $L := S^3 \cap \{f - 1\}_\Gamma(0)$, take $\xi_L = T_L \cap J_0(T_L)$. $S^3$'s contact structure descends to $S^3/\Gamma$, recall: $\alpha_0 = i_2(ud\bar{u} - \bar{u}u + vd\bar{v} - \bar{v}v)$. Lemma (N) $(S^3/\Gamma, \xi_{S^3/\Gamma})$ is contactomorphic to $(L, \xi_L)$. Topology of link tells us nature of singularity... Are there dynamical implications?
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Lemma (N) $(\mathbb{S}^3/\Gamma, \xi_{\mathbb{S}^3/\Gamma})$ is contactomorphic to $(\mathcal{L}, \xi_\mathcal{L})$. 

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$(S^3/\Gamma, \xi_{S^3/\Gamma})$ is contactomorphic to $(L, \xi_L)$.
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A fortuitous dynamical relationship

\[ \alpha' = (1 + \epsilon h^* H) \alpha_0 \]
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\[ R' = \frac{1}{(1+\epsilon h^* H)} R_0 + \frac{\epsilon}{(1+\epsilon h^* H)^2} \tilde{X}_H. \]
A fortuitous dynamical relationship

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\begin{align*}
S^3 & \xrightarrow{h} S^2 \xrightarrow{H} \mathbb{R} \\
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X_H &= J_0 \nabla H,
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- $X_H = J_0 \nabla H$, use symmetry of $\Gamma$ to pick $H$
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\[
S^3 \\
\downarrow h \\
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For \( \Gamma = \mathbb{T}^* (E_6\text{-type}) \) take \( H = xyz. \)
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\[ h \downarrow \]
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Reeb orbits which generate chain complex correspond to presentation of \( S^3/\Gamma \) as a Seifert fiber space!
Other Seifert fiber spaces
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Connections to Chen-Ruan orbifold homology and string topology
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Precise nature of relationship with symplectic homology
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- Extending work to hold in more generality in dimension 3
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Extending work to hold in more generality in dimension 3
Look at dimensions $> 3$??
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Other dynamical questions involving contact structures
Thanks!